DISCRETE SERIES WHITTAKER FUNCTIONS ON $Spin(2n, 2)$

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Abstract. Discrete series Whittaker functions on $Spin(2n, 2)$ are studied. The dimensions of the space of both algebraic and continuous Whittaker modelss are explicitly determined. They are described by a sum of dimensions of irreducible representations of $Spin(2n - 3, 2)$. Also obtained are the Mellin-Barnes type integral formulas of the Whittaker functions associated with minimal K-type vectors.

1. Introduction

Let $G_{\mathbb{R}}$ be a real semisimple Lie group and $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ be its Iwasawa decomposition. Let η be a one dimensional unitary representation of $N_{\mathbb{R}}$. Given a representation π of $G_{\mathbb{R}}$, a realization of π in the induced representation $\mathrm{Ind}_{N_{\mathbb{R}}}^{G_{\mathbb{R}}}\eta$ is called a Whittaker model of π. By this realization, a vector v in π is expressed by a function on $G_{\mathbb{R}}$, which we call the Whittaker function associated with v.

Mainly from the number theoretical point of view, Takayuki Oda and his colleagues have calculated explicit formulas of Whittaker functions and generalized spherical functions on various low rank groups and of various representations.

Besides the number theoretical interest, the theory of Whittaker models are also interesting from the analytic points of view. It is well known that, in the case of the principal series of $SL(2, \mathbb{R})$, the Whittaker functions associated with a $K_{\mathbb{R}}$-type vector is expressed by the classical Whittaker's confluent hypergeometric function. In the case of representations of higher rank groups, the image of a Whittaker model gives an example of multi-variable confluent hypergeometric functions. Therefore, we may expect that we can study such functions by means of representation theory.

Another interesting aspect is the relationship with the invariants of representations. First, let us consider algebraic Whittaker models, namely $(\mathfrak{g}, K_{\mathbb{R}})$-module intertwining operators from a Harish-Chandra $(\mathfrak{g}, K_{\mathbb{R}})$-module π to $\mathrm{Ind}_{N_{\mathbb{R}}}^{G_{\mathbb{R}}}\eta$. If the one dimensional unitary representation η of $N_{\mathbb{R}}$ is non-degenerate (cf.§3.1), we can judge the existence of non-trivial Whittaker models by the Gelfand-Kirillov dimension of π, and the dimension of the Whittaker models is given by the Bernstein degree of it. Secondly, let us consider continuous Whittaker models, namely continuous intertwining operators from the C^∞-globalization π_∞ of a $(\mathfrak{g}, K_{\mathbb{R}})$-module π to the C^∞-induced space C^∞-$\mathrm{Ind}_{N_{\mathbb{R}}}^{G_{\mathbb{R}}}\eta$. We can judge the existence of non-trivial continuous Whittaker models by the wave front set of π. If π is a discrete series, the dimension of continuous Whittaker models are expressed by the Bernstein degree and geometric data of nilpotent orbits. These results, due to H. Matumoto ([9], [10]), are summarized in §3.

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The author thinks that we can observe the relationship between the invariants of representations and the structure of Whittaker models more clearly when we treat Whittaker models of non-quasi-split groups than that of quasi-split groups. Suppose $G_{\mathbb{R}}$ is a non-quasi-split real semisimple Lie group. Consider a discrete series representation π which has non-trivial Whittaker models. Then it is observed that the degree of an irreducible component of the associated variety corresponds to the dimension of the solution space of one system of differential equation. It is also observed that the multiplicity of the associated cycle corresponds to the dimension of the right $Z_{M_{\mathbb{R}}}(\eta)$-module structure of the solution space of the gradient type differential-difference equation $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$. (For the definition of this equation, see §3.4). Here, $M_{\mathbb{R}}(\eta)$ is the centralizer of η in $M_{\mathbb{R}} = Z_{K_{\mathbb{R}}}(A_{\mathbb{R}})$. This observation is obtained in a former paper [13] of the author's. He expects that such correspondence holds for general higher rank group cases, and hopes to explain it in a natural way.

Until now, there are not so many concrete examples of non-quasi-split cases. For such reasons, we investigates the case when $G_{\mathbb{R}}$ is the non-quasi-split group $Spin(2n,2)$, π is a discrete series of $G_{\mathbb{R}}$, and η is a non-degenerate character of $N_{\mathbb{R}}$ in this paper. This setting is the same as in [6], except for the group $G_{\mathbb{R}}$. Since $Spin(4,2) \simeq SU(2,2)$, this paper is a generalization of [6].

The sections are organized as follows. In §2, we briefly review the structure of $Spin(2n,2)$ and parametrize its discrete series representations. The set of discrete series representations of $Spin(2n,2)$ is divided into $2n+2$ parts $\Xi_{m,\pm}$, $m = 1,\ldots,n+1$. §3 is mainly devoted to the presentation of general theory. We review H. Matumoto's results on the existence and the dimension of Whittaker models in §3.1, J. T. Chang's results on the associated cycles of discrete series in §3.2, and H. Yamashita's results on the realization of algebraic Whittaker models of discrete series in §3.4. In §3.3, we apply Chang's theory to our $Spin(2n,2)$ case. As a result, combined with Theorem 5.13, we obtain the dimension of the space of algebraic Whittaker models:

Theorem 1.1 (Corollary 3.6, Theorem 3.8)**.** *Suppose $G_{\mathbb{R}} = Spin(2n,2)$.*

(1) *The discrete series π_Λ has non-trivial algebraic Whittaker models if and only if $\Lambda \in \Xi_{m,\pm}$, $m = 2,\ldots,n$.*
(2) *The dimension of the space of algebraic Whittaker models of π_Λ, $\Lambda \in \Xi_{m,\pm}$, $m = 2,\ldots,n$, is*

$$4 \sum_{\substack{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-2} \geq \mu_{m-2} \geq \lambda_{m-1} \\ \lambda_m \geq \mu'_1 \geq \lambda_{m+1} \geq \cdots \geq \lambda_{n-1} \geq \mu'_{n-m} \geq |\lambda_n|}} \dim V^{Spin(2n-3,\mathbb{C})}_{(\mu_1,\ldots,\mu_{m-2},\mu'_1,\ldots,\mu'_{n-m})}.$$

In §4, we write down the gradient type differential-difference equations, which describe discrete series Whittaker functions, by using Gelfand-Tsetlin basis of the minimal $K_{\mathbb{R}}$-type. After that, we put them in order. Solving such differential-difference equations reduces to getting a coefficient function of some special minimal $K_{\mathbb{R}}$-type vector, which we call "corner vector". This reduction is discussed in §4.4, and the result is as follows.

Theorem 1.2 (Theorem 4.23)**.** *Let Q^+_{n-1} be the Gelfand-Tsetlin pattern defined in Definition 4.22. Let $\phi(a) = \sum_{Q \in GT(\lambda)} c(Q;a)Q$ be a solution of the differential-difference equation $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$, which describes an algebraic Whittaker model of the discrete series π^*_Λ, $\Lambda \in \Xi_{m,\pm}$, $m = 2,\ldots,n$. If $c(Q^+_{n-1};a)$ is known, then it determines all the $c(Q;a)$ for $Q \in GT(\lambda)$ containing the same $\mathbf{q}_{2n-4}$ part as Q^+_{n-1}.*

By this theorem, we may restrict our interest to determine the coefficient function $c(Q_{n-1}^{+};a)$. In §5, we deduce a system of differential equations satisfied by the coefficient function $c(Q;a)$ for some special Gelfand-Tsetlin pattern Q. The set of such special patterns contains Q_{n-1}^{+}. Also obtained are the Mellin-Barnes type integral expression of these functions. In order to present these results briefly, we use the notation defined in §§4, 5. See (4.21) for the definition of $K_6(m')$, and see §5.2 for the definition of α_j, β_j, t_j, $n(Q,m';a)$.

Theorem 1.3 (Proposition 5.8, Proposition 5.9)**.** *Assume $Q \in GT(\lambda)$ satisfies* (4.20)*,* (4.23) *and* (4.24)*.*

(1) *Define $f(Q,m';t) := n(Q,m';a)^{-1}c(Q;a)$. Then, $f(Q,m';t)$ is a solution of*

$$(\partial_{t_1}^2 - \partial_{t_2}^2 - t_1^2) f(Q,m';t) = 0,$$
$$\left\{ \prod_{p=1}^{N_2} (\partial_{t_2} + \alpha_p) + \frac{t_2}{t_1}(\partial_{t_1} - \partial_{t_2}) \prod_{p=3}^{N_2} (\partial_{t_2} + \beta_p) \right\} f(Q,m';t) = 0.$$

(2) *Let C_j be a loop starting and ending at $+\infty$, crossing the real axis at $-\alpha_j - 1 < s < -\alpha_j$, and encircling all poles of $\Gamma(-\alpha_p - s)$, $p = j, \dots, N_2$, once in the negative direction, but none of the poles of $\Gamma(\beta_p + s)$, $p = 3, \dots, j$. Define*

$$\begin{Bmatrix} f_1^K(Q,m';t) \\ f_1^I(Q,m';t) \end{Bmatrix} := \frac{1}{2\pi\iota} \int_{C_1} \frac{\prod_{p=1}^{N_2} \Gamma(-\alpha_p - s)}{\prod_{p=3}^{N_2} \Gamma(1 - \beta_p - s)} \begin{Bmatrix} t_2^s \, K_{-s}(t_1) \\ (-t_2)^s \, I_{-s}(t_1) \end{Bmatrix} ds,$$

and, for $j = 2, \dots, N_2$, define

$$\begin{Bmatrix} f_j^K(Q,m';t) \\ f_j^I(Q,m';t) \end{Bmatrix}$$
$$:= \frac{1}{2\pi\iota} \int_{C_j} \frac{\prod_{p=j}^{N_2} \Gamma(-\alpha_p - s) \prod_{p=3}^{j} \Gamma(\beta_p + s)}{\prod_{p=1}^{j-1} \Gamma(1 + \alpha_p + s) \prod_{p=j+1}^{N_2} \Gamma(1 - \beta_p - s)} \begin{Bmatrix} (-t_2)^s \, K_{-s}(t_1) \\ t_2^s \, I_{-s}(t_1) \end{Bmatrix} ds.$$

Here, $K_\mu(z)$ and $I_\nu(z)$ are modified Bessel functions, and $\iota = \sqrt{-1}$. These integrals absolutely converge in $\mathbb{C}^2 \setminus (\{t_1 = 0\} \cup \{t_2 = 0\})$, and they form a basis of the solution space of the system of differential equations in (1).

An interesting fact is that not all of these functions generates a solution of the whole differential-difference equation $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$. In §5.3, we construct shift operators satisfied by the functions f_j^K, f_j^I (Propositions 5.10, 5.12). By these shift operators, we know that only 4 in the $2N_2$ functions f_j^K, f_j^I *do* generate solutions of the whole differential-difference equations.

Theorem 1.4 (Theorem 5.13)**.** *Let $\Lambda \in \Xi_{m,\pm}$, $m = 2, \dots, n$. Then the functions $f_j^L(Q_{n-1}^{+}, n-1; t)$, with $j = 1, 2$, $L = K, I$, completely determines the solutions of $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$.*

Some of the above solutions may define a continuous intertwining operator from the C^∞-globalization $(\pi_\Lambda^*)_\infty$ to $C^\infty(G_\mathbb{R}/N_\mathbb{R};\eta)$.

Theorem 1.5 (Theorem 5.15)**.** *Let $\Lambda \in \Xi_{m,\pm}$, $m = 2, \dots, n$. Suppose the character η of $N_\mathbb{R}$ is the one defined by* (4.6)*. If $\mp\eta_2 > 0$, then the dimension of the space of*

continuous intertwining operators is

$$\sum_{\substack{\lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\lambda_{m-2}\geq\mu_{m-2}\geq\lambda_{m-1}\\ \lambda_m\geq\mu'_1\geq\lambda_{m+1}\geq\cdots\geq\lambda_{n-1}\geq\mu'_{n-m}\geq|\lambda_n|}} \dim V^{Spin(2n-3,\mathbb{C})}_{(\mu_1,\dots,\mu_{m-2},\mu'_1,\dots,\mu'_{n-m})}.$$

Each continuous intertwining operator corresponds to f_1^K defined in Theorem 1.3 (2). On the other hand, if $\mp\eta_2 < 0$, then this space is zero.

When $G_{\mathbb{R}} = Spin(2n+1,2)$, we have similar results on the Whittaker models of discrete series representations. Details are discussed elsewhere about this case.

Before ending the introduction, we define some notation. Suppose $H_{\mathbb{R}}$ is a real Lie group. Write $\mathfrak{h}_0$ the Lie algebra of $H_{\mathbb{R}}$, $\mathfrak{h}$ the complexification of $\mathfrak{h}_0$ and H a complexification of $H_{\mathbb{R}}$. This notation will be applied to groups denoted by other Roman letters in the same way without comment. For two integers $a<b$, let $[a,b]$ be the interval $\{x\in\mathbb{Z} | a\leq x\leq b\}$. The imaginary unit is denoted by ι.

2. The group $Spin(2n,2)$ and its discrete series

2.1. **Structure of** $Spin(2n,2)$**.** Let $G_{\mathbb{R}} = Spin(2n,2) \subset Spin(2n+2,\mathbb{C})$ $(n\geq 2)$ be the connected two-fold linear cover of $SO_0(2n,2)$, whose maximal compact subgroup $K_{\mathbb{R}}$ is isomorphic to $Spin(2n)\times Spin(2)$. We realize the Lie algebra $\mathfrak{g}_0 = \mathfrak{so}(2n,2)$ as

$$\mathfrak{so}(2n,2) = \left\{ \begin{pmatrix} X_{11} & \iota X_{12} \\ -\iota\, {}^t X_{12} & X_{22}\end{pmatrix} \middle| \begin{matrix} X_{11}\in \mathrm{Alt}_{2n}(\mathbb{R}), X_{22}\in\mathrm{Alt}_2(\mathbb{R}), \\ X_{12}\in M_{2n,2}(\mathbb{R})\end{matrix} \right\}.$$

Let $\theta X = -{}^t\bar{X}$ be a Cartan involution of $\mathfrak{g}_0$ and let $\mathfrak{g}_0 = \mathfrak{k}_0+\mathfrak{p}_0$ the corresponding Cartan decomposition. Denote elementary matrices by $E_{ij} = (\delta_{ki}\delta_{lj})_{k,l=1,\dots,2n+2}$ and set $F_{ij} := E_{ij}-E_{ji}$. Then

$$\{F_{j,k} | 1\leq k<j\leq 2n, \text{ or } (j,k) = (2n+2,2n+1)\} \quad \text{and}$$
$$\{\iota F_{2n+j,k} | 1\leq j\leq 2, 1\leq k\leq 2n\}$$

are bases of $\mathfrak{k}_0$ and $\mathfrak{p}_0$ respectively.

Let

$$A_k := \iota F_{2n+k,2n-2+k}, \qquad \mathfrak{a}_0 := \mathbb{R}A_1+\mathbb{R}A_2,$$

and define $f_j\in\mathfrak{a}_0^*$ by $f_j(A_k) = \delta_{j,k}$. Then $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. The restricted root system $\Delta(\mathfrak{g}_0,\mathfrak{a}_0)$ is

$$\Delta(\mathfrak{g}_0,\mathfrak{a}_0) = \{\pm f_1\pm f_2, \pm f_1, \pm f_2\}.$$

Choose a positive system

$$\Delta^+(\mathfrak{g}_0,\mathfrak{a}_0) = \{\pm f_1+f_2, f_1, f_2\},$$

and denote the corresponding nilpotent subalgebra $\sum_{\alpha\in\Delta^+(\mathfrak{g}_0,\mathfrak{a}_0)}(\mathfrak{g}_0)_\alpha$ by $\mathfrak{n}_0$. Here $(\mathfrak{g}_0)_\alpha$ is the root space corresponding to a root α. One obtains an Iwasawa decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0+\mathfrak{a}_0+\mathfrak{n}_0, \qquad\qquad G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}},$$

where $A_{\mathbb{R}} = \exp\mathfrak{a}_0$ and $N_{\mathbb{R}} = \exp\mathfrak{n}_0$. Let

$$X^k_{f_j} := F_{2n-2+j,k}+\iota F_{2n+j,k}, \qquad (1\leq j\leq 2, 1\leq k\leq 2n-2),$$
$$X_{f_1+f_2} := F_{2n,2n-1}-\iota F_{2n+1,2n}+\iota F_{2n+2,2n-1}-F_{2n+2,2n+1},$$
$$X_{-f_1+f_2} := F_{2n,2n-1}+\iota F_{2n+1,2n}+\iota F_{2n+2,2n-1}+F_{2n+2,2n+1}.$$

Then $\{X^k_{f_j} | 1 \le k \le 2n-2\}$ is a basis of $(\mathfrak{g}_0)_{f_j}$, and $\{X_{\pm f_1+f_2}\}$ is a basis of $(\mathfrak{g}_0)_{\pm f_1+f_2}$.

2.2. **Parameterization of discrete series.** Let us now parametrize the discrete series of $G_{\mathbb{R}}$. Take a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ defined by

$$T_k := -\iota F_{2k,2k-1}, \qquad \mathfrak{t} := \bigoplus_{k=1}^{n+1} \mathbb{C} T_k.$$

Let $\{e_j | 1 \le j \le n+1\}$ be the dual of $\{T_k\}$. The root systems $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ and $\Delta_c := \Delta(\mathfrak{k}, \mathfrak{t})$ are

$$\Delta = \{\pm e_i \pm e_j | 1 \le i < j \le n+1\}, \qquad \Delta_c = \{\pm e_i \pm e_j | 1 \le i < j \le n\},$$

respectively. Choose a positive system

$$\Delta_c^+ = \{e_i \pm e_j | 1 \le i < j \le n\}$$

of Δ_c. There are $2n+2$ positive systems $\Delta^+_{1,\pm}, \ldots, \Delta^+_{n+1,\pm}$ of Δ containing Δ_c^+, defined by

$$\begin{aligned}
\Delta^+_{m,+} &= \Delta^+_c \cup \{e_i \pm e_{n+1} | 1 \le i \le m-1\} \cup \{e_{n+1} \pm e_i | m \le i \le n\}, \\
&\quad \text{if} \quad 1 \le m \le n+1, \\
\Delta^+_{m,-} &= \Delta^+_c \cup \{e_i \pm e_{n+1} | 1 \le i \le m-1\} \cup \{-e_{n+1} \pm e_i | m \le i \le n\}, \\
&\quad \text{if} \quad 1 \le m \le n, \\
\Delta^+_{n+1,-} &= \Delta^+_c \cup \{e_i \pm e_{n+1} | 1 \le i \le n-1\} \cup \{-e_n \pm e_{n+1}\}.
\end{aligned}$$

The discrete series representations of $G_{\mathbb{R}}$ are parametrized by Harish-Chandra parameters. By definition, the set of Harish-Chandra parameters Ξ^+ is

$$\Xi^+ = \{\Lambda \in \mathfrak{t}^* | \Lambda \text{ is } \Delta\text{-regular, } \Delta_c^+\text{-dominant, and } \Lambda + \rho \text{ is } K_{\mathbb{R}}\text{-analytically integral}\}.$$

Here, ρ is half the sum of positive roots for some positive system of Δ. Hereafter, we denote by π_Λ the discrete series representation corresponding to $\Lambda \in \Xi^+$. In our case, Ξ^+ is divided into $2n+2$ parts:

$$\Xi^+ = \bigcup_{m=1}^{n+1} \Xi_{m,+} \cup \bigcup_{m=1}^{n+1} \Xi_{m,-}, \qquad \Xi_{m,\pm} := \{\Lambda \in \Xi^+ | \langle \Lambda, \beta \rangle \ge 0, \forall \beta \in \Delta^+_{m,\pm}\}.$$

3. Discrete series Whittaker functions

3.1. **Whittaker models.** Let $\eta : N_{\mathbb{R}} \longrightarrow \mathbb{C}^\times$ be a non-degenerate unitary character of $N_{\mathbb{R}}$. We denote the differential representation $\mathfrak{n}_0 \to \iota\mathbb{R}$ of η by the same letter η. "Non-degenerate" means that η is non-trivial on every root space corresponding to a simple root of $\Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)$. The space of analytic and smooth Whittaker functions are defined by

$$\mathcal{A}(G_{\mathbb{R}}/N_{\mathbb{R}}; \eta) = \{F : G_{\mathbb{R}} \overset{\text{analytic}}{\to} \mathbb{C} | F(gn) = \eta(n)^{-1} F(g), \text{ for } n \in N_{\mathbb{R}}, g \in G_{\mathbb{R}}\},$$

$$C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}}; \eta) = \{F : G_{\mathbb{R}} \overset{C^\infty}{\to} \mathbb{C} | F(gn) = \eta(n)^{-1} F(g), \text{ for } n \in N_{\mathbb{R}}, g \in G_{\mathbb{R}}\},$$

respectively. These are representation spaces of $G_{\mathbb{R}}$ by left translation.

Let (π, V) be a representation of $G_{\mathbb{R}}$ or a Harish-Chandra $(\mathfrak{g}, K_{\mathbb{R}})$-module. A realization of (π, V) in $\mathcal{A}(G_{\mathbb{R}}/N_{\mathbb{R}}; \eta)$ is called a Whittaker model of (π, V). By this realization, a vector $v \in V$ is expressed by a function on $G_{\mathbb{R}}$, which we call a Whittaker function associated with v.

We review some results, due to Matumoto, on the existence and the dimension of Whittaker models.

Theorem 3.1 ([9])**.** *Let V be an irreducible Harish-Chandra $(\mathfrak{g}, K_{\mathbb{R}})$-module.*

(1) *V has a non-trivial Whittaker model if and only if the Gelfand-Kirillov dimension* $\mathrm{Dim}V$ *of V is equal to* $\dim \mathfrak{n}_0$.
(2) *If V has a non-trivial Whittaker model, then the dimension of Whittaker models is equal to the Bernstein degree* $\mathrm{Deg}V$ *of V.*

Any irreducible Harish-Chandra $(\mathfrak{g}, K_{\mathbb{R}})$-module V admits an infinitesimal character. Therefore, any smooth Whittaker function $\psi(v)(g)$, with $v \in V$ and $\psi \in \mathrm{Hom}_{\mathfrak{g},K_{\mathbb{R}}}(V, C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$, is an eigenfunction of the Casimir operator. Since the Casimir operator is an elliptic differential operator, $\psi(v)$ is real analytic. Therefore we may identify $\mathrm{Hom}_{\mathfrak{g},K_{\mathbb{R}}}(V, \mathcal{A}(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$ with $\mathrm{Hom}_{\mathfrak{g},K_{\mathbb{R}}}(V, C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$, if V is irreducible.

The next problem is to specify the continuous intertwining operators. For a Harish-Chandra $(\mathfrak{g}, K_{\mathbb{R}})$-module (π, V), let (π_∞, V_∞) be its C^∞-globalization. This is a Fréchet representation of $G_{\mathbb{R}}$. The space of continuous intertwining operators from V_∞ to $C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta)$ will be denoted by $\mathrm{Hom}^\infty_{G_{\mathbb{R}}}(V_\infty, C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$. This is a subspace of $\mathrm{Hom}_{\mathfrak{g},K_{\mathbb{R}}}(V, C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$, and it is isomorphic to the space

$$\mathrm{Wh}^\infty_{-\eta}(V) = \{\xi \in V'_\infty | \pi'_\infty(X)\xi = -\eta(X)\xi, X \in \mathfrak{n}_0\}$$

of $C^{-\infty}$-Whittaker vectors (cf [10]). Here, (π'_∞, V'_∞) is the continuous dual to (π_∞, V_∞) with respect to the $U(\mathfrak{g})$-topology.

The next theorem, also due to Matumoto, presents a condition for the existence of non-trivial continuous intertwining operators and the dimension of the space of such operators. In order to state his results, we introduce some conventions. Since a unitary character ψ is determined by the value on the root spaces corresponding to simple roots, we may regard ψ as an element of $\iota(\mathfrak{n}_0/[\mathfrak{n}_0,\mathfrak{n}_0])^* \subset \iota\mathfrak{g}_0^*$. Using the Killing form, we identify $\mathfrak{g}_0^*$ with $\mathfrak{g}_0$. Note that ψ is non-degenerate if and only if $\iota^{-1}\mathrm{Ad}(G_{\mathbb{R}})\psi \subset \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is a principal nilpotent $G_{\mathbb{R}}$-orbit. We denote by $\mathcal{P}(G_{\mathbb{R}})$ the set of principal nilpotent $G_{\mathbb{R}}$-orbits. The wave front set of V is denoted by $\mathrm{WF}(V)$. For the definition of wave front set, see [10].

Theorem 3.2 ([10])**.** *Let $G_{\mathbb{R}}$ be a connected real reductive linear Lie group and let V be an irreducible Harish-Chandra $(\mathfrak{g}, K_{\mathbb{R}})$-module. Let ψ be a non-degenerate unitary character on $\mathfrak{n}_0$.*

(1) $\mathrm{Wh}^\infty_\psi(V)$ *is non-zero if and only if* $\psi \in \mathrm{WF}(V)$.
(2) *If V is the Harish-Chandra module of a discrete series representation and* $\psi \in \mathrm{WF}(V)$*, then*

$$\dim \mathrm{Wh}^\infty_\psi(V) = \frac{\#\mathcal{P}(G_{\mathbb{R}})}{\#W_{G_{\mathbb{R}}}}\mathrm{Deg}V,$$

where $W_{G_{\mathbb{R}}}$ is the little Weyl group of $G_{\mathbb{R}}$.

3.2. **Chang's results.** Let us recall J. T. Chang's papers [2], [3], in which associated cycles of discrete series are determined for some cases.

Let V be a Harish-Chandra $(\mathfrak{g}, K_{\mathbb{R}})$-module. Choose a $K_{\mathbb{R}}$-stable finite dimensional generating subspace V_0 of V, and define a filtration $V_n := U(\mathfrak{g})_n V_0$ of V. Here $\{U(\mathfrak{g})_n | n = 0, 1, 2, \dots\}$ is the standard filtration of $U(\mathfrak{g})$. By this filtration, $M := \mathrm{gr}V$ admits an $S(\mathfrak{g}) \simeq \mathrm{gr}U(\mathfrak{g})$-module structure. The associated variety

$\mathcal{AV}(V)$ of V is the support $\mathrm{Supp}M \subset \mathfrak{g}^*$ of M. Note that $\mathcal{AV}(V)$ is a closure of finite K-orbits on $\mathfrak{p}$. Let $X_1, \ldots, X_k$ be the irreducible components of $\mathcal{AV}(V)$, and let m_i be the length of $S(\mathfrak{g})_{\mathfrak{P}_i}$ module $M_{\mathfrak{P}_i}$, where $\mathfrak{P}_i$ is the minimal prime ideal corresponding to the irreducible variety X_i. The associated cycle $\mathcal{AC}(V)$ of V is the formal sum $\sum_i m_i X_i$.

Assume that $G_{\mathbb{R}}$ has discrete series. Then $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$. Let $\mathfrak{b} = \mathfrak{t} + \bar{\mathfrak{u}}$ be a Borel subalgebra of $\mathfrak{g}$, and let $B \subset G$ be the corresponding Borel subgroup. This Borel determines a positive root system Δ^+ of $\Delta(\mathfrak{g}, \mathfrak{t})$ so that $\bar{\mathfrak{u}}$ corresponds to negative roots. Let $\Delta_c^+ \subset \Delta^+$ be the set of compact positive roots and denote $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, $\rho_c = \frac{1}{2}\sum_{\alpha\in\Delta_c^+}\alpha$, $\rho_n = \rho - \rho_c$. We may and do regard $\mathfrak{b}$ as a point on the flag variety $X := G/B$. By the choice of $\mathfrak{b}$, the K-orbit $Z := K \cdot \mathfrak{b} \simeq K/K \cap B$ is closed. Here, "dot" means the adjoint action.

Suppose $\Lambda \in \Xi^+$ is Δ^+-dominant. Let τ be the character of $T := \exp \mathfrak{t}$ with $d\tau = \Lambda - \rho$. This character gives rise to a K-homogeneous line bundle on Z, and we denote by $\mathcal{L}_{\Lambda-\rho}$ its sheaf of local sections on Z. Let $j : Z \to X$ be the embedding map, and let $j_+(\mathcal{L}_{\Lambda-\rho})$ be the direct image in the category of $\mathcal{D}$-modules. It is well known that the global section space $\Gamma(X, j_+(\mathcal{L}_{\Lambda-\rho}))$ realizes the Harish-Chandra module of π_Λ. Note that the lowest $K_{\mathbb{R}}$-type sheaf for π_Λ is $\mathcal{L}_{\Lambda-\rho} \otimes \det \mathcal{N}_{Z|X} \simeq \mathcal{L}_{\Lambda-\rho_c+\rho_n}$, where $\mathcal{N}_{Z|X}$ is the normal sheaf of Z in X.

Let $T^*X \simeq G \times_B (\mathfrak{g}/\mathfrak{b})^* \simeq G \times_B \bar{\mathfrak{u}}$ be the cotangent bundle on X. Here, we identify $(\mathfrak{g}/\mathfrak{b})^*$ with $\bar{\mathfrak{u}}$ by a fixed invariant bilinear form on $\mathfrak{g}$. By this identification and $TZ \simeq K \times_{K\cap B} (\mathfrak{k}/\mathfrak{k} \cap \mathfrak{b})$, the conormal bundle T_Z^*X is isomorphic to $K \times_{K\cap B} (\bar{\mathfrak{u}} \cap \mathfrak{p})$. For each point $x \in X$, representing a Borel subalgebra $\mathfrak{b}_x$, the fiber T_x^*X in T^*X is given by $(\mathfrak{g}/\mathfrak{b}_x)^* \subset \mathfrak{g}^*$. The moment map $\gamma : T^*X \to \mathfrak{g}^*$ is given by, on each fiber, the inclusion $(\mathfrak{g}/\mathfrak{b}_x)^* \hookrightarrow \mathfrak{g}^*$. For the discrete series π_Λ, the associated variety $\mathcal{AV}(\pi_\Lambda)$ is a closure of a single K-orbit on $\mathfrak{p}$, and it coincides with the moment map image $\gamma(T_Z^*X) \simeq K \cdot (\bar{\mathfrak{u}} \cap \mathfrak{p})$. Therefore, the associated cycle $\mathcal{AC}(\pi_\Lambda)$ of π_Λ is an integral multiple of $\gamma(T_Z^*X)$.

Theorem 3.3 ([3]). *Let ξ be a generic point of $\bar{\mathfrak{u}} \cap \mathfrak{p} \subset \gamma(T_Z^*X)$. Then the associated cycle of the discrete series representation π_Λ is*

$$\mathcal{AC}(\pi_\Lambda) = \dim H^0(\gamma^{-1}(\xi), \mathcal{L}_{\Lambda-\rho_c+\rho_n}|_{\gamma^{-1}(\xi)})\, \gamma(T_Z^*X).$$

Under some condition, the explicit formula of this coefficient can be obtained. Let $N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p}) := \{k \in K | k \cdot \xi \in \bar{\mathfrak{u}} \cap \mathfrak{p}\}$. Then it is not hard to show that $\gamma^{-1}(\xi) \simeq N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p})^{-1} \cdot \mathfrak{b} \subset Z$. Note that $N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p})$ is not a group. In order to calculate $N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p})$, let us consider the following groups. Let S be the set of all compact simple roots, $\langle S \rangle$ the root system generated by S. Let $\mathfrak{l} := \mathfrak{t} + \sum_{\alpha\in\langle S\rangle} \mathfrak{g}_\alpha$, and let $\mathfrak{q} = \mathfrak{l} + \bar{\mathfrak{v}} \supset \mathfrak{b}$ be the parabolic subalgebra whose Levi part is $\mathfrak{l}$. The analytic subgroups with Lie algebras $\mathfrak{l}$, $\mathfrak{q}$ are denoted by L, Q respectively.

Theorem 3.4 ([3]). *Let $K(\xi)$ be the centralizer of ξ in K and $K(\xi)_r^0$ be the identity component of the reductive part of $K(\xi)$. If*

$$N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p}) = (K \cap Q)K(\xi) = (K \cap Q)K(\xi)_r^0, \tag{3.1}$$

then

$$H^0(\gamma^{-1}(\xi), \mathcal{L}_{\Lambda-\rho_c+\rho_n}|_{\gamma^{-1}(\xi)}) \simeq \text{Coh-Ind} \uparrow_{K(\xi)_r^0 \cap Q}^{K(\xi)_r^0} \text{Res} \downarrow_{K(\xi)_r^0 \cap Q}^{K\cap Q} V_{\Lambda-\rho_c+\rho_n}^{K\cap Q}.$$

Here Coh-Ind *is the cohomological induction, i.e. the operation which makes the irreducible representation of the large group with the same highest weight as that of the small group.*

3.3. $Spin(2n,2)$ **case.** Let us apply the general theory to our $Spin(2n,2)$ case.

Proposition 3.5. (1) *If* $\Lambda \in \Xi_{1,\pm}$*, then* $\mathrm{Dim}(\pi_\Lambda) = 2n$*.*
(2) *If* $\Lambda \in \Xi_{m,\pm}$*,* $m = 2, \dots, n$*, then* $\mathrm{Dim}(\pi_\Lambda) = 4n-2 = \dim \mathfrak{n}_0$*.*
(3) *If* $\Lambda \in \Xi_{n+1,\pm}$*, then* $\mathrm{Dim}(\pi_\Lambda) = 4n-3$*.*

Proof. Let $X_\alpha \in \mathfrak{g}_\alpha$ be a non-zero root vector for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$. For each case, we can choose

$$\xi = \begin{cases} (1) & X_{\pm e_n \mp e_{n+1}} + X_{\mp e_n \mp e_{n+1}} \\ (2) & X_{-e_1 \pm e_{n+1}} + X_{e_n \mp e_{n+1}} + X_{-e_n \mp e_{n+1}} \\ (3) & X_{\mp e_n \pm e_{n+1}} + X_{-e_{n-1} \mp e_{n+1}} \end{cases}$$

as a generic point of $\bar{\mathfrak{u}} \cap \mathfrak{p}$. It is not hard to calculate the centralizer $K(\xi)$, and finally we obtain $\mathrm{Dim}(\pi_\Lambda) = \dim(K \cdot \xi) = \dim K - \dim K(\xi) = 2n$, $4n-2$ and $4n-3$, respectively. □

Corollary 3.6. *The discrete series* π_Λ *has a non-trivial algebraic Whittaker model if and only if* $\Lambda \in \Xi_{m,\pm}$*,* $m = 2, \dots, n$*.*

Chang has shown that the condition (3.1) is satisfied if $G_\mathbb{R}$ is a connected real rank one group. It is also satisfied in our case.

Proposition 3.7. *The condition* (3.1) *is satisfied when* $\Lambda \in \Xi_{m,\pm}$*,* $m = 2, \dots, n$*.*

Proof. Suppose $\Lambda \in \Xi_{m,\pm}$, $m = 2, \dots, n$. In this case,

$$\bar{\mathfrak{u}} \cap \mathfrak{p} = \bigoplus_{i=1}^{m-1} (\mathfrak{g}_{-e_i \pm e_{n+1}} \oplus \mathfrak{g}_{-e_i \mp e_{n+1}}) \oplus \bigoplus_{i=m}^{n} (\mathfrak{g}_{e_i \mp e_{n+1}} \oplus \mathfrak{g}_{-e_i \mp e_{n+1}}).$$

We choose a generic point ξ of $\bar{\mathfrak{n}} \cap \mathfrak{p}$ as in the proof of Proposition 3.5. For this ξ,

$$K(\xi)_r \simeq \{\pm 1\} \times Spin(2n-3, \mathbb{C}),$$
$$K(\xi) = K(\xi)_r \exp\left(\sum_{i=2}^{n-1} (a_i X_{-e_1+e_i} + b_i X_{-e_1-e_i}) + a_n (X_{-e_1+e_n} + X_{-e_1-e_n}) \right),$$

where X_α's are appropriately chosen non-zero root vectors. For $\Lambda \in \Xi_{m,\pm}$, the parabolic subgroup $K \cap Q$ corresponds to the set of simple roots $\{e_1 - e_2, \dots, e_{m-2} - e_{m-1}, e_m - e_{m+1}, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$. Let $Q_1 \supset K \cap B$ be the parabolic subgroup of K, whose Levi subalgebra corresponds to the set of simple roots $\{e_2 - e_3, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$. Note that Q_1 contains $K(\xi)$. Let $k = qwq_1$ be the Bruhat decomposition of $k \in N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p})$ with respect to $(K \cap Q)\backslash K/Q_1$, where $q \in K \cap Q$, $w \in (\mathfrak{S}_{m-1} \times (\mathfrak{S}_{n-m+1} \ltimes \mathbb{Z}_2^{n-m}))\backslash \mathfrak{S}_n \ltimes \mathbb{Z}_2^{n-1}/(\mathfrak{S}_{n-1} \ltimes \mathbb{Z}_2^{n-2})$, $q_1 \in Q_1$. Since $K \cap Q$ normalizes $\bar{\mathfrak{u}} \cap \mathfrak{p}$, $k \cdot \xi \in \bar{\mathfrak{u}} \cap \mathfrak{p}$ is equivalent to $q_1 \cdot \xi \in w^{-1}(\bar{\mathfrak{u}} \cap \mathfrak{p})$. By direct computation, this is true only if $w = e$, and we can show that, if $q_1 \cdot \xi \in \bar{\mathfrak{u}} \cap \mathfrak{p}$, then $q_1 \in (K \cap Q)K(\xi)$. Thus we get $N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p}) \subset (K \cap Q)K(\xi)$. Since the inverse inclusion is trivial, we get $N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p}) = (K \cap Q)K(\xi)$. Finally, since each connected component of $K(\xi)$ meets $\exp \mathfrak{t} \subset K \cap Q$ and the unipotent radical of $K(\xi)$ is contained in $K \cap Q$, we know $N_K(\xi, \bar{\mathfrak{u}} \cap \mathfrak{p}) = (K \cap Q)K(\xi) = (K \cap Q)K(\xi)_r^0$. □

Let V_ν^F be the irreducible finite dimensional representation of a reductive group F with the highest weight ν. For $\Lambda \in \Xi_{m,\pm}$, $m = 2, \dots, n$, let $\tilde{\lambda} = (\lambda; \lambda_{n+1}) = (\lambda_1, \dots, \lambda_n; \lambda_{n+1}) = \Lambda - \rho_c + \rho_n$ be the Blattner parameter of π_Λ. Since

$$\begin{aligned} K(\xi)_r^0 &\simeq Spin(2n-3, \mathbb{C}), \\ K \cap Q &\simeq GL(m-1, \mathbb{C}) \times Spin(2(n-m+1), \mathbb{C}) \times Spin(2, \mathbb{C}) \\ &\qquad \ltimes (\text{unipotent radical}), \\ K(\xi)_r^0 \cap Q &\simeq GL(m-2, \mathbb{C}) \times Spin(2n-2m+1, \mathbb{C}) \ltimes (\text{unipotent radical}), \end{aligned}$$

we have

$$\begin{aligned} & V_{\tilde{\lambda}}^{K\cap Q} = V_{(\lambda_1,\dots,\lambda_{m-1})}^{GL(m-1,\mathbb{C})} \boxtimes V_{(\lambda_m,\dots,\lambda_n)}^{Spin(2(n-m+1),\mathbb{C})} \boxtimes V_{\lambda_{n+1}}^{Spin(2,\mathbb{C})}, \\ & \operatorname{Res}\downarrow_{K(\xi)_r^0\cap Q}^{K\cap Q} V_{\tilde{\lambda}}^{K\cap Q} \\ & \quad \simeq \bigoplus_{\substack{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-2} \geq \mu_{m-2} \geq \lambda_{m-1} \\ \lambda_m \geq \mu'_1 \geq \lambda_{m+1} \geq \cdots \geq \lambda_{n-1} \geq \mu'_{n-k} \geq |\lambda_n|}} V_{(\mu_1,\dots,\mu_{m-2})}^{GL(m-2,\mathbb{C})} \boxtimes V_{(\mu'_1,\dots,\mu'_{n-m})}^{Spin(2n-2m+1,\mathbb{C})}, \\ & \text{Coh-Ind}\uparrow_{K(\xi)_r^0\cap Q}^{K(\xi)_r^0} \operatorname{Res}\downarrow_{K(\xi)_r^0\cap Q}^{K\cap Q} V_{\tilde{\lambda}}^{K\cap Q} \\ & \quad \simeq \bigoplus_{\substack{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-2} \geq \mu_{m-2} \geq \lambda_{m-1} \\ \lambda_m \geq \mu'_1 \geq \lambda_{m+1} \geq \cdots \geq \lambda_{n-1} \geq \mu'_{n-m} \geq |\lambda_n|}} V_{(\mu_1,\dots,\mu_{m-2},\mu'_1,\dots,\mu'_{n-m})}^{Spin(2n-3,\mathbb{C})}. \end{aligned}$$

Theorem 3.8. *The Bernstein degree of π_Λ, $\Lambda \in \Xi_{m,\pm}$, $m = 2, \dots, n$, is*

$$C \sum_{\substack{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-2} \geq \mu_{m-2} \geq \lambda_{m-1} \\ \lambda_m \geq \mu'_1 \geq \lambda_{m+1} \geq \cdots \geq \lambda_{n-1} \geq \mu'_{n-m} \geq |\lambda_n|}} \dim V_{(\mu_1,\dots,\mu_{m-2},\mu'_1,\dots,\mu'_{n-m})}^{Spin(2n-3,\mathbb{C})},$$

where C is a general constant independent of Λ.

3.4. **Realization of Whittaker functions.** Embedding of a discrete series into an induced representation is realized by gradient type differential-difference equations. We review Yamashita's results (cf. [16]).

Let η be a non-degenerate unitary character of $N_\mathbb{R}$. For a finite dimensional representation (τ, V_τ) of $K_\mathbb{R}$, define

$$\begin{aligned} & C_\tau^\infty(K_\mathbb{R} \backslash G_\mathbb{R} / N_\mathbb{R}; \eta) \\ & := \{F : G_\mathbb{R} \xrightarrow{C^\infty} V_\tau | F(kgn) = \eta(n)^{-1} \tau(k) F(g), \text{ for } n \in N_\mathbb{R}, g \in G_\mathbb{R}, k \in K_\mathbb{R}\}. \end{aligned}$$

Let, as before, $\tilde{\lambda} = \Lambda - \rho_c + \rho_n$ be the Blattner parameter corresponding to a Harish-Chandra parameter Λ. Let $(\tau_{\tilde{\lambda}}, V_{\tilde{\lambda}})$ be the irreducible finite dimensional representation of $K_\mathbb{R}$ with the highest weight $\tilde{\lambda}$. Let $(\mathrm{Ad}, \mathfrak{p})$ be the adjoint representation of $K_\mathbb{R}$ on $\mathfrak{p}$.

Fix an invariant bilinear form $\langle \ , \ \rangle$ on $\mathfrak{g}_0$ and choose an orthonormal basis $\{X_i\}$ of $\mathfrak{p}_0$. Define a differential-difference operator $\nabla_{\tilde{\lambda},\eta}$ by

$$\begin{aligned} & \nabla_{\tilde{\lambda},\eta} : C_{\tau_{\tilde{\lambda}}}^\infty(K_\mathbb{R} \backslash G_\mathbb{R} / N_\mathbb{R}; \eta) \to C_{\tau_{\tilde{\lambda}} \otimes \mathrm{Ad}}^\infty(K_\mathbb{R} \backslash G_\mathbb{R} / N_\mathbb{R}; \eta), \\ & \nabla_{\tilde{\lambda},\eta} \phi(g) := \sum_i L_{X_i} \phi(g) \otimes X_i. \end{aligned}$$

Here, L_{X_i} is the left translation.

Let Δ_n^+ be the set of non-compact roots α with $\langle \alpha, \Lambda \rangle > 0$. Then the irreducible decomposition of $\tau_{\tilde{\lambda}} \otimes \mathrm{Ad}$ is $\oplus_{\alpha \in \Delta_n^+ \cup (-\Delta_n^+)} m_\alpha \tau_{\tilde{\lambda}+\alpha}$, $m_\alpha \in \{0, 1\}$. Let $\tau_{\tilde{\lambda}}^- := \oplus_{\alpha \in \Delta_n^+} m_{-\alpha} \tau_{\tilde{\lambda}-\alpha}$ be the negative part and let $\mathrm{pr}^- : \tau_{\tilde{\lambda}} \otimes \mathrm{Ad} \to \tau_{\tilde{\lambda}}^-$ be the natural projection. Define a differential-difference operator $\mathcal{D}_{\tilde{\lambda},\eta}$ by

$$\mathcal{D}_{\tilde{\lambda},\eta} := \mathrm{pr}^- \circ \nabla_{\tilde{\lambda},\eta} : C^\infty_{\tau_{\tilde{\lambda}}}(K_\mathbb{R} \backslash G_\mathbb{R} / N_\mathbb{R}; \eta) \to C^\infty_{\tau_{\tilde{\lambda}}^-}(K_\mathbb{R} \backslash G_\mathbb{R} / N_\mathbb{R}; \eta).$$

Theorem 3.9 ([16])**.** *Let π_Λ^* be the dual Harish-Chandra module of π_Λ. If the Blattner $\tilde{\lambda}$ of π_Λ is far from the walls, then*

$$\mathrm{Hom}_{\mathfrak{g}, K_\mathbb{R}}(\pi_\Lambda^*, C^\infty(G_\mathbb{R}/N_\mathbb{R}; \eta)) \simeq \mathrm{Ker}(\mathcal{D}_{\tilde{\lambda},\eta}).$$

Remark 3.10. Suppose $G_\mathbb{R} = Spin(2n, 2)$ and $\Lambda = \sum_{i=1}^{n+1} \Lambda_i e_i$. The Harish-Chandra parameter of π_Λ^* is $\Lambda^* := \sum_{i=1}^{n-1} \Lambda_i e_i + (-1)^n \Lambda_n e_n - \Lambda_{n+1} e_{n+1}$. Therefore, if $\Lambda \in \Xi_{m,\pm}$ and $m \leq n$, then $\Lambda^* \in \Xi_{m,\mp}$. The Blattner parameter $\tilde{\lambda}$ corresponding to $\Lambda = \sum_{i=1}^{n+1} \Lambda_i e_i \in \Xi_{m,\pm}$, $m \leq n$, is

$$\tilde{\lambda} = \sum_{i=1}^{m-1} (\Lambda_i - n + i + 1) e_i + \sum_{i=m}^{n} (\Lambda_i - n + i) e_i + (\Lambda_{n+1} \pm (n - m + 1)) e_{n+1}.$$

Especially, the n-th components λ_n, λ_n^* of the Blattner parameters of π_Λ, π_Λ^* are Λ_n, $(-1)^n \Lambda_n$, respectively. Therefore, if $\Lambda \in \Xi_{m,\pm}$, $m = 2, \ldots, n$, then the Bernstein degrees of π_Λ and π_Λ^* are identical because of Theorem 3.8.

4. Radial $A_\mathbb{R}$ part of $\mathcal{D}_{\tilde{\lambda},\eta}$

In this section, we write down the differential-difference equation $\mathcal{D}_{\tilde{\lambda},\eta} \phi = 0$ explicitly. After that, we reduce our computation to getting a coefficient function of some special vector.

4.1. **Irreducible decomposition of tensor product representation.** The Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(2n, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$ and, as a vector space, $\mathfrak{p}$ is isomorphic to the matrix space $M_{2n,2}(\mathbb{C})$. The adjoint representation $(\mathrm{ad}, \mathfrak{p})$ of $\mathfrak{k}$ on $\mathfrak{p}$ is

$$\begin{aligned} &\mathfrak{so}(2n, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C}) \curvearrowright M_{2n,2}(\mathbb{C}), \\ &(A, B) \cdot X = AX - XB, \quad \text{for } A \in \mathfrak{so}(2n, \mathbb{C}), B \in \mathfrak{so}(2, \mathbb{C}), X \in M_{2n,2}(\mathbb{C}). \end{aligned}$$

Let $(\tau^k, \mathbb{C}^k)$ be the natural representation of $\mathfrak{so}(k, \mathbb{C})$ on $\mathbb{C}^k$. By the identification $\mathbb{C}^{2n} \otimes \mathbb{C}^2 \simeq M_{2n,2}(\mathbb{C})$, $u \otimes v \mapsto u\,{}^t v$,

$$(\mathrm{ad}, \mathfrak{p}) \simeq (\tau^{2n}, \mathbb{C}^{2n}) \boxtimes (\tau^2, \mathbb{C}^2).$$

Note that $\iota F_{2n+i,j} \in \mathfrak{p}$ corresponds to the vector $v_j^{2n} \boxtimes v_i^2 \in \tau^{2n} \boxtimes \tau^2$, where $v_i^k = {}^t(0, \ldots 0, \overset{i}{1}, 0, \ldots, 0)$ is a standard basis of $\mathbb{C}^k$.

We realize the representation $\tau_{\tilde{\lambda}}$ of $K_\mathbb{R}$ by using the Gelfand-Tsetlin basis.

Definition 4.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a dominant integral weight of $Spin(2n)$. A *(λ-)Gelfand-Tsetlin pattern* is a set of vectors $Q = (\mathbf{q}_1, \ldots, \mathbf{q}_{2n-1})$ such that

(1) $\mathbf{q}_i = (q_{i,1}, q_{i,2}, \ldots, q_{i,\lfloor (i+1)/2 \rfloor})$.
(2) The numbers $q_{i,j}$ are all integers or all half integers.
(3) $q_{2i+1,j} \geq q_{2i,j} \geq q_{2i+1,j+1}$, for any $j = 1, \ldots, i-1$.
(4) $q_{2i+1,i} \geq q_{2i,i} \geq |q_{2i+1,i+1}|$.
(5) $q_{2i,j} \geq q_{2i-1,j} \geq q_{2i,j+1}$, for any $j = 1, \ldots, i-1$.

(6) $q_{2i,i} \geq q_{2i-1,i} \geq -q_{2i,i}$.
(7) $q_{2n-1,j} = \lambda_j$.

Here, $\lfloor a \rfloor$ is the largest integer not greater than a. The set of all λ-Gelfand-Tsetlin patterns is denoted by $GT(\lambda)$.

Notation 4.2. For any set or number $*$ depending on $Q \in GT(\lambda)$, we denote it by $*(Q)$, if we need to specify Q. For example, $q_{i,j}(Q)$ is the $q_{i,j}$ part of $Q \in GT(\lambda)$.

Theorem 4.3 ([5])**.** *Let λ be a dominant integral weight of $Spin(2n)$ and let $(\tau_\lambda, V_\lambda^{Spin(2n)})$ be the irreducible representation of $Spin(2n)$ with the highest weight λ. Then $GT(\lambda)$ is a basis of $(\tau_\lambda, V_\lambda^{Spin(2n)})$.*

The action of elements $F_{p,q} \in \mathfrak{so}(2n)$ is expressed as follows. For $j > 0$, let

$$l_{2i-1,j} := q_{2i-1,j} + i - j, \qquad l_{2i-1,-j} := -l_{2i-1,j},$$
$$l_{2i,j} := q_{2i,j} + i + 1 - j, \qquad l_{2i,-j} := -l_{2i,j} + 1,$$

and let $l_{2i,0} = 0$. Define $a_{p,q}(Q)$ by

$$a_{2i-1,j}(Q) = \mathrm{sgn} j \sqrt{-\frac{\prod_{1 \leq |k| \leq i-1}(l_{2i-1,j} + l_{2i-2,k}) \prod_{1 \leq |k| \leq i}(l_{2i-1,j} + l_{2i,k})}{4 \prod_{\substack{1 \leq |k| \leq i, \\ k \neq \pm j}} (l_{2i-1,j} + l_{2i-1,k})(l_{2i-1,j} + l_{2i-1,k} + 1)}},$$

for $j = \pm 1, \ldots, \pm i$, and

$$a_{2i,j}(Q) = \epsilon_{2i,j}(Q) \sqrt{-\frac{\prod_{1 \leq |k| \leq i}(l_{2i,j} + l_{2i-1,k}) \prod_{1 \leq |k| \leq i+1}(l_{2i,j} + l_{2i+1,k})}{(4l_{2i,j}^2 - 1) \prod_{\substack{0 \leq |k| \leq i \\ k \neq \pm j}} (l_{2i,j} + l_{2i,k})(l_{2i,j} - l_{2i,k})}},$$

for $j = 0, \pm 1, \ldots, \pm i$, where $\epsilon_{2i,j}(Q)$ is $\mathrm{sgn} j$ if $j \neq 0$, and $\mathrm{sgn}(q_{2i-1,i} q_{2i+1,i+1})$ if $j = 0$.

Let $\sigma_{a,b}$ be the shift operator, sending $\mathbf{q}_a$ to $\mathbf{q}_a + (0, \ldots, \overset{|b|}{\mathrm{sgn}(b)}, 0, \ldots, 0)$. For notational convenience, we often write $\tau_{i,j} := \sigma_{2n-3,i}\sigma_{2n-2,j}$.

Theorem 4.4 ([5])**.** *Under the above notation, the Lie algebra action is expressed as*

$$\tau_\lambda(F_{2i+1,2i})Q = \sum_{1 \leq |j| \leq i} a_{2i-1,j}(Q) \sigma_{2i-1,j} Q,$$
$$\tau_\lambda(F_{2i+2,2i+1})Q = \sum_{0 \leq |j| \leq i} a_{2i,j}(Q) \sigma_{2i,j} Q.$$

In the following of this paper, we assume that the Blattner parameter $\tilde{\lambda}$ is far from the walls. It follows that the numbers λ_j satisfy

$$\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > |\lambda_n| \tag{4.1}$$

and the differences between adjacent numbers are sufficiently large.

Let $e_k = (0, \ldots, \overset{k}{1}, 0, \ldots, 0) \in \mathbb{C}^n$ and $e_{-k} := -e_k$. Let

$$\mathrm{pr}_k : V_\lambda^{Spin(2n)} \otimes \mathbb{C}^{2n} \to V_{\lambda+e_k}^{Spin(2n)}, \quad \text{for} \quad k = \pm 1, \ldots, \pm n$$

be the projection operator along the irreducible decomposition

$$V_\lambda^{Spin(2n)} \otimes \mathbb{C}^{2n} \simeq \bigoplus_{1 \leq |k| \leq n} V_{\lambda+e_k}^{Spin(2n)}.$$

In order to describe this operator explicitly, we identify $Q \in GT(\lambda)$ with the Gelfand-Tsetlin pattern $(Q, \mathbf{q}_{2n})$ of a representation of $Spin(2n+1)$, where $\mathbf{q}_{2n} := (\lambda_1 + 1, \ldots, \lambda_{n-1} + 1, |\lambda_n| + 1)$. Just the same way as in the proof of [8, Proposition 4.3] and [13, Lemma 3.1.3], we get the following formulas.

Lemma 4.5. *For $Q \in GT(\lambda)$ and $k = \pm 1, \ldots, \pm n$,*

$$\mathrm{pr}_k(Q \otimes v_{2n}^{2n}) = a_{2n-1,k}(Q)\sigma_{2n-1,k}Q, \tag{4.2}$$

$$\mathrm{pr}_k(Q \otimes v_{2n-1}^{2n}) = \sum_{0 \le |j| \le n-1} \frac{a_{2n-2,j}(Q)a_{2n-1,k}(\tau_{0,j}Q)}{l_{2n-2,j} - l_{2n-1,k}} \tau_{0,j}\sigma_{2n-1,k}Q, \tag{4.3}$$

$$\begin{aligned} &\mathrm{pr}_k(Q \otimes v_{2n-2}^{2n}) \\ &\quad = \sum_{1 \le |i| \le n-1} \sum_{0 \le |j| \le n-1} \frac{a_{2n-3,i}(Q)a_{2n-2,j}(\tau_{i,0}Q)a_{2n-1,k}(\tau_{i,j}Q)}{(l_{2n-3,i} - l_{2n-2,j} + 1)(l_{2n-2,j} - l_{2n-1,k})} \tau_{i,j}\sigma_{2n-1,k}Q. \end{aligned} \tag{4.4}$$

Remark 4.6. For $Q \in GT(\lambda)$, it is not hard to see that $a_{2i-1,j}(Q) = 0$ if and only if $\tau_{0,j}Q \notin GT(\lambda)$, and that $a_{2i,j}(Q) = 0$ if and only if either (i) $j \neq 0$ and $\sigma_{2i,j}Q \notin GT(\lambda)$ or (ii) $j = 0$ and $q_{2i-1,i}$ or $q_{2i+1,i+1} = 0$. Moreover,

(1) if $j, k \neq 0$, the coefficient of $\tau_{0,j}\sigma_{2n-1,k}Q$ in (4.3) is non-zero if and only if $\tau_{0,j}\sigma_{2n-1,k}Q \in GT(\lambda + e_k)$, and
(2) if $i, j, k \neq 0$, the coefficient of $\tau_{i,j}\sigma_{2n-1,k}Q$ in (4.4) is non-zero if and only if $\tau_{i,j}\sigma_{2n-1,k}Q \in GT(\lambda + e_k)$.

We know that $V_{\tilde{\lambda}}^{K_\mathbb{R}} = V_\lambda^{Spin(2n)} \boxtimes V_{\lambda_{n+1}}^{Spin(2)}$ and $\mathfrak{p} \simeq \mathbb{C}^{2n} \boxtimes \mathbb{C}^2$. Therefore, the irreducible decomposition of $V_{\tilde{\lambda}}^{K_\mathbb{R}} \otimes \mathfrak{p}$ is

$$V_{\tilde{\lambda}}^{K_\mathbb{R}} \otimes \mathfrak{p} \simeq \bigoplus_{1 \le |k| \le n} V_{\lambda+e_k}^{Spin(2n)} \boxtimes V_{\lambda_{n+1}+1}^{Spin(2)} \oplus \bigoplus_{1 \le |k| \le n} V_{\lambda+e_k}^{Spin(2n)} \boxtimes V_{\lambda_{n+1}-1}^{Spin(2)}. \tag{4.5}$$

Since the irreducible representations of $Spin(2)$ are one dimensional, we identify $Q \in GT(\lambda)$ with a vector of $V_{\tilde{\lambda}}^{K_\mathbb{R}}$, on which T_{n+1} acts by the scalar λ_{n+1}. Such an identification is also applied to $V_{\lambda+e_k}^{Spin(2n)} \boxtimes V_{\lambda_{n+1}\pm 1}^{Spin(2)}$. More precisely, for $Q \in GT(\lambda)$, regard $Q_k^\pm := \sigma_{2n-1,k}Q$ as a vector in $V_{\lambda+e_k}^{Spin(2n)} \boxtimes V_{\lambda_{n+1}\pm 1}^{Spin(2)}$, on which T_{n+1} acts by the scalar $\lambda_{n+1} \pm 1$.

Let $\mathrm{pr}_{k,\pm} = \mathrm{pr}_k \boxtimes \mathrm{pr}_\pm$ be the projection operator from $V_{\tilde{\lambda}}^{K_\mathbb{R}} \otimes \mathfrak{p}$ to $V_{\lambda+e_k}^{Spin(2n)} \boxtimes V_{\lambda_{n+1}\pm 1}^{Spin(2)}$ along the irreducible decomposition (4.5). Then, by normalizing vectors appropriately, we have the following explicit formulas from Lemma 4.5.

Lemma 4.7. *For $Q \in GT(\lambda)$ and $k = \pm 1, \ldots, \pm n$,*

$$\mathrm{pr}_{k,\pm}(Q \otimes (v_{2n}^{2n} \boxtimes v_2^2)) = a_{2n-1,k}(Q)Q_k^\pm,$$

$$\mathrm{pr}_{k,\pm}(Q \otimes (v_{2n-1}^{2n} \boxtimes v_2^2)) = \sum_{0 \le |j| \le n-1} \frac{a_{2n-2,j}(Q)a_{2n-1,k}(\tau_{0,j}Q)}{l_{2n-2,j} - l_{2n-1,k}} \tau_{0,j}Q_k^\pm,$$

$$\begin{aligned} &\mathrm{pr}_{k,\pm}(Q \otimes (v_{2n-2}^{2n} \boxtimes v_2^2)) \\ &\quad = \sum_{1 \le |i| \le n-1} \sum_{0 \le |j| \le n-1} \frac{a_{2n-3,i}(Q)a_{2n-2,j}(\tau_{i,0}Q)a_{2n-1,k}(\tau_{i,j}Q)}{(l_{2n-3,i} - l_{2n-2,j} + 1)(l_{2n-2,j} - l_{2n-1,k})} \tau_{i,j}Q_k^\pm, \end{aligned}$$

$$\mathrm{pr}_{k,\pm}(Q \otimes (v_j^{2n} \boxtimes v_1^2)) = \mp\iota\,\mathrm{pr}_{k,\pm}(Q \otimes (v_j^{2n} \boxtimes v_2^2)) \quad \textit{for} \quad j = 2n, 2n-1, 2n-2.$$

The actions of Casimir elements of $\mathfrak{so}(k) \subset \mathfrak{so}(2n)$ on the Gelfand-Tsetlin bases are as follows. Let $C_k := \sum_{1\le j<i\le k} F_{ij}^2$. Since this is a constant multiple of the Casimir element of $\mathfrak{so}(k)$, it acts on an irreducible representation of $\mathfrak{so}(k)$ by a scalar. Especially it acts on $Q \in GT(\lambda)$ by a scalar, since Q is contained in the $V^{Spin(k)}_{\mathbf{q}_{k-1}}$-isotropic subspace. This scalar is calculated in [13, §5.1].

Lemma 4.8. *Let* $2\rho_k := (k-2, k-4, \ldots, k-2\lfloor k/2\rfloor)$. *For* $Q \in GT(\lambda)$ *and* $k = 2, \ldots, 2n$,

$$\tau_\lambda(C_k)Q = -(|\mathbf{q}_{k-1}|^2 + 2\langle \mathbf{q}_{k-1}, \rho_k\rangle)Q.$$

4.2. **Differential-difference equation $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$.** Under some appropriate normalization of the invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}_0$, $\{\iota F_{2n+i,j} | 1 \le i \le 2, 1 \le j \le 2n\}$ forms an orthonormal basis of $\mathfrak{p}_0$. The Iwasawa decompositions of these vectors are given by

$$\begin{aligned}
&\iota F_{2n+i,k} = X^k_{f_i} - F_{2n-2+i,k}, \quad \text{for} \quad 1 \le i \le 2, 1 \le k \le 2n-2,\\
&\iota F_{2n+i,2n-2+i} = A_i, \quad \text{for} \quad 1 \le i \le 2,\\
&\iota F_{2n+1,2n} = \frac{1}{2}(X_{-f_1+f_2} - X_{f_1+f_2}) - \iota T_{n+1},\\
&\iota F_{2n+2,2n-1} = \frac{1}{2}(X_{-f_1+f_2} + X_{f_1+f_2}) - F_{2n,2n-1}.
\end{aligned}$$

Let $M_\mathbb{R}$ be the centralizer of $A_\mathbb{R}$ in $K_\mathbb{R}$. Since $M_\mathbb{R}$ acts on $\mathfrak{n}_0/[\mathfrak{n}_0, \mathfrak{n}_0]$, it acts on the set of unitary characters of $N_\mathbb{R}$. Therefore, when we calculate $\mathrm{Ker}(\mathcal{D}_{\tilde{\lambda},\eta})$, we may choose "manageable" non-degenerate unitary character η in its $M_\mathbb{R}$-orbit. Let $M_\mathbb{R}(\eta)$ be the centralizer of η in $M_\mathbb{R}$. Since $M_\mathbb{R}(\eta)$ acts on $\mathrm{Ker}(\mathcal{D}_{\tilde{\lambda},\eta})$ by the right translation, the space of Whittaker models has $M_\mathbb{R}(\eta)$-module structure.

As a "manageable" character, we choose η satisfying

$$\eta(X^k_{f_i}) = \begin{cases} \iota\eta_1 & \text{if } (i,k) = (1, 2n-2), \\ 0 & \text{else,} \end{cases} \qquad \eta(X_{-f_1+f_2}) = \iota\eta_2, \quad \eta(X_{f_1+f_2}) = 0 \tag{4.6}$$

with $\eta_1 > 0, \eta_2 \neq 0$. Here, we denote the differential of η by the same symbol η.

Because of the Iwasawa decomposition, an element of $C^\infty_{\tau_{\tilde{\lambda}}}(K_\mathbb{R}\backslash G_\mathbb{R}/N_\mathbb{R}; \eta)$ is determined by its restriction to $A_\mathbb{R}$. Thus, we consider the restriction of $\phi \in C^\infty_{\tau_{\tilde{\lambda}}}(K_\mathbb{R}\backslash G_\mathbb{R}/N_\mathbb{R}; \eta)$ to $A_\mathbb{R}$.

Introduce a coordinate in $A_\mathbb{R}$ by

$$(\mathbb{R}_{>0})^2 \ni (a_1, a_2) \mapsto \exp((\log a_1)A_1 + (\log a_2)A_2) \in A_\mathbb{R}$$

and define $\partial_i := a_i\partial/\partial a_i$. The left action of Lie algebra elements on ϕ is as follows:

$$\begin{aligned}
&\text{If } X \in \mathfrak{k}_0, \text{ then} \quad L_X\phi(a) = \frac{d}{dt}\Big|_{t=0} \phi(\exp(-tX)a) = -d\tau_{\tilde{\lambda}}(X)\phi(a),\\
&\text{if } X \in \mathfrak{n}_0, \text{ then} \quad L_X\phi(a) = \frac{d}{dt}\Big|_{t=0} \phi(a\exp(-t\mathrm{Ad}(a^{-1})X)) = \eta(\mathrm{Ad}(a^{-1})X)\phi(a),\\
&\text{and for } A_i, \quad L_{A_i}\phi(a) = \frac{d}{dt}\Big|_{t=0} \phi(\exp(-tA_i)a) = -\partial_i\phi(a).
\end{aligned}$$

Since $\{\iota F_{2n+i,j} | 1 \le i \le 2, 1 \le j \le 2n\}$ is an orthonormal basis of $\mathfrak{p}_0$,

$$
\begin{aligned}
&\nabla_{\tilde{\lambda},\eta}\phi \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2n} L_{\iota F_{2n+i,j}}\phi \otimes \iota F_{2n+i,j} \\
&= \sum_{i=1}^{2}\sum_{k=1}^{2n-2} L_{X^k_{f_i} - F_{2n-2+i,k}}\phi \otimes \iota F_{2n+i,k} + L_{(X_{-f_1+f_2} - X_{f_1+f_2})/2 - \iota T_{n+1}}\phi \otimes \iota F_{2n+1,2n} \\
&\quad + \sum_{i=1}^{2} L_{A_i}\phi \otimes \iota F_{2n+i,2n-2+i} + L_{(X_{-f_1+f_2} + X_{f_1+f_2})/2 - F_{2n,2n-1}}\phi \otimes \iota F_{2n+2,2n-1} \\
&= -\sum_{i=1}^{2} \partial_i\phi \otimes \iota F_{2n+i,2n-2+i} + \iota\frac{\eta_1}{a_1}\phi \otimes \iota F_{2n+1,2n-2} \\
&\quad + \iota\frac{\eta_2 a_1}{2a_2}(\phi \otimes \iota F_{2n+2,2n-1} + \phi \otimes \iota F_{2n+1,2n}) \\
&\quad + \sum_{i=1}^{2}\sum_{k=1}^{2n-3+i} d\tau_{\tilde{\lambda}}(F_{2n-2+i,k})\phi \otimes \iota F_{2n+i,k} + \iota d\tau_{\tilde{\lambda}}(T_{n+1})\phi \otimes \iota F_{2n+1,2n}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&(d\tau_{\tilde{\lambda}} \otimes \mathrm{ad})(F_{2n-2+i,k})^2(v \otimes \iota F_{2n+i,2n-2+i}) \\
&= d\tau_{\tilde{\lambda}}(F_{2n-2+i,k})^2 v \otimes \iota F_{2n+i,2n-2+i} \\
&\quad - 2d\tau_{\tilde{\lambda}}(F_{2n-2+i,k})v \otimes \iota F_{2n+i,k} - v \otimes \iota F_{2n+i,2n-2+i}
\end{aligned}
$$

for $v \in V_{\tilde{\lambda}}^{K_{\mathbb{R}}}$, we have

$$
\begin{aligned}
&d\tau_{\tilde{\lambda}}(F_{2n-2+i,k})v \otimes \iota F_{2n+i,k} \\
&= \frac{1}{2}\{(d\tau_{\tilde{\lambda}} \otimes 1)(F_{2n-2+i,k})^2 - (d\tau_{\tilde{\lambda}} \otimes \mathrm{ad})(F_{2n-2+i,k})^2 - 1\}(v \otimes \iota F_{2n+i,2n-2+i}).
\end{aligned}
$$

Here, 1 is the trivial representation of $\mathfrak{k}$. Since $\iota F_{2n+i,j}$ corresponds to $v_j^{2n} \boxtimes v_i^2$ under $\mathfrak{p} \simeq \mathbb{C}^{2n} \boxtimes \mathbb{C}^2$ and $\sum_{k=1}^{2n-3+i} F_{2n-2+i,k}^2 = C_{2n-2+i} - C_{2n-3+i}$, we have

(4.7)

$$
\begin{aligned}
\nabla_{\tilde{\lambda},\eta}\phi = &\frac{1}{2}\sum_{i=1}^{2}\{-2\partial_i + (d\tau_{\tilde{\lambda}} \otimes 1)(C_{2n-2+i} - C_{2n-3+i}) \\
&\qquad - (d\tau_{\tilde{\lambda}} \otimes \mathrm{ad})(C_{2n-2+i} - C_{2n-3+i}) - 2n + 3 - i\} \\
&\qquad \times \phi \otimes (v_{2n-2+i}^{2n} \boxtimes v_i^2) + \iota\lambda_{n+1}\phi \otimes (v_{2n}^{2n} \boxtimes v_1^2) \\
&+ \iota\frac{\eta_1}{a_1}\phi \otimes (v_{2n-2}^{2n} \boxtimes v_1^2) + \iota\frac{\eta_2 a_1}{2a_2}(\phi \otimes (v_{2n-1}^{2n} \boxtimes v_2^2) + \phi \otimes (v_{2n}^{2n} \boxtimes v_1^2)).
\end{aligned}
$$

Let us calculate the projection of $\nabla_{\tilde{\lambda},\eta}\phi$ to each irreducible component of $V_{\tilde{\lambda}}^{K_{\mathbb{R}}} \otimes \mathfrak{p}$. By the identification $GT(\lambda) \subset V_{\tilde{\lambda}}^{K_{\mathbb{R}}}$ explained in §4.1, we write

$$\phi(a) = \sum_{Q \in GT(\lambda)} c(Q; a)Q.$$

We need to calculate the projection

$$
\begin{aligned}
\tag{4.8}
&\mathrm{pr}_k(\{(d\tau_{\tilde{\lambda}} \otimes 1)(C_{2n-2+i} - C_{2n-3+i}) \\
&\qquad - (d\tau_{\tilde{\lambda}} \otimes \mathrm{ad})(C_{2n-2+i} - C_{2n-3+i})\}Q \otimes v^{2n}_{2n-2+i}) \\
&= \mathrm{pr}_k((d\tau_{\tilde{\lambda}} \otimes 1)(C_{2n-2+i} - C_{2n-3+i})Q \otimes v^{2n}_{2n-2+i}) \\
&\qquad - d\tau_{\lambda+e_k}(C_{2n-2+i} - C_{2n-3+i})\mathrm{pr}_k(Q \otimes v^{2n}_{2n-2+i}).
\end{aligned}
$$

When $i = 2$, this is a multiple of $a_{2n-1,k}(Q)\sigma_{2n-1,k}Q$. By (4.2) and Lemma 4.8, its coefficient is

$$
\begin{aligned}
&-(|\mathbf{q}_{2n-1}|^2 + 2\langle \mathbf{q}_{2n-1}, \rho_{2n}\rangle) + (|\mathbf{q}_{2n-2}|^2 + 2\langle \mathbf{q}_{2n-2}, \rho_{2n-1}\rangle) \\
&\qquad + (|\mathbf{q}_{2n-1} + e_k|^2 + 2\langle \mathbf{q}_{2n-1} + e_k, \rho_{2n}\rangle) - (|\mathbf{q}_{2n-2}|^2 + 2\langle \mathbf{q}_{2n-2}, \rho_{2n-1}\rangle) \\
&= 2(\mathrm{sgn} k)\lambda_{|k|} + 1 + (\mathrm{sgn} k)(2n - 2|k|) \\
&= 2l_{2n-1,k} + 1.
\end{aligned}
$$

Analogously, when $i = 1$, (4.8) is a linear combination of

$$
\frac{a_{2n-2,j}(Q)a_{2n-1,k}(\sigma_{2n-2,j}Q)}{l_{2n-2,j} - l_{2n-1,k}}\sigma_{2n-2,j}\sigma_{2n-1,k}Q,
$$

whose coefficient is $2l_{2n-2,j}$.

By these and Lemma 4.7, the projection of (4.7) to $V^{Spin(2n)}_{\lambda+e_k} \boxtimes V^{Spin(2)}_{\lambda_{n+1}\pm 1}$ is

$$
\begin{aligned}
\tag{4.9}
&\mathrm{pr}_{k,\pm}(\nabla_{\tilde{\lambda},\eta}\phi) \\
&= -\sum_{Q \in GT(\lambda)} a_{2n-1,k}(Q)\left(\partial_2 - l_{2n-1,k} + n - 1 \mp \lambda_{n+1} \mp \frac{\eta_2 a_1}{2a_2}\right) c(Q;a)Q^{\pm}_k \\
&\quad \pm \iota \sum_{Q \in GT(\lambda)} \sum_{0 \le |j| \le n-1} \frac{a_{2n-2,j}(Q)a_{2n-1,k}(\tau_{0,j}Q)}{l_{2n-2,j} - l_{2n-1,k}} \\
&\qquad\qquad \times \left(\partial_1 - l_{2n-2,j} + n - 1 \pm \frac{\eta_2 a_1}{2a_2}\right) c(Q;a)\tau_{0,j}Q^{\pm}_k \\
&\quad \pm \frac{\eta_1}{a_1} \sum_{Q \in GT(\lambda)} \sum_{1 \le |i| \le n-1} \sum_{0 \le |j| \le n-1} \\
&\qquad\qquad \times \frac{a_{2n-3,i}(Q)a_{2n-2,j}(\tau_{i,0}Q)a_{2n-1,k}(\tau_{i,j}Q)}{(l_{2n-3,i} - l_{2n-2,j} + 1)(l_{2n-2,j} - l_{2n-1,k})} c(Q;a)\tau_{i,j}Q^{\pm}_k.
\end{aligned}
$$

In order to rewrite (4.9) in a simple form, we need to know the zero points of coefficients in the right hand. By Remark 4.6, we have the following lemma:

Lemma 4.9. *Suppose $Q \in GT(\lambda)$.*

1. *For $k = \pm 1, \dots, \pm n$, $a_{2n-1,k}(Q)$ is not zero if and only if $\sigma_{2n-1,k}Q \in GT(\lambda + e_k)$.*
2. *For $k = \pm 1, \dots, \pm n$ and $j = \pm 1, \dots, \pm(n-1)$,*

$$
\frac{a_{2n-2,j}(Q)a_{2n-1,k}(\tau_{0,j}Q)}{l_{2n-2,j} - l_{2n-1,k}} = \frac{a_{2n-2,j}(\sigma_{2n-1,k}Q)a_{2n-1,k}(Q)}{l_{2n-2,j} - l_{2n-1,k} - 1}
$$

is not zero if and only if $\tau_{0,j}\sigma_{2n-1,k}Q \in GT(\lambda + e_k)$.

(3) *For $k=\pm 1,\dots,\pm n$, $j=\pm 1,\dots,\pm(n-1)$ and $i=\pm 1,\dots,\pm(n-1)$,*

$$\frac{a_{2n-3,i}(Q)a_{2n-2,j}(\tau_{i,0}Q)a_{2n-1,k}(\tau_{i,j}Q)}{(l_{2n-3,i}-l_{2n-2,j}+1)(l_{2n-2,j}-l_{2n-1,k})}$$
$$=\frac{a_{2n-3,i}(\tau_{0,j}Q)a_{2n-2,j}(\sigma_{2n-1,k}Q)a_{2n-1,k}(Q)}{(l_{2n-3,i}-l_{2n-2,j})(l_{2n-2,j}-l_{2n-1,k}-1)}$$

is not zero if and only if $\tau_{i,j}\sigma_{2n-1,k}Q\in GT(\lambda+e_k)$.

By this lemma, we can write down (4.9) in a simple form.

Proposition 4.10. *For $k=\pm 1,\dots,\pm n$, $\mathrm{pr}_{k,\pm}(\nabla_{\tilde{\lambda},\eta}\phi)=0$ is equivalent to the following equations.*

(1) *If $Q\in GT(\lambda)$ and $\sigma_{2n-1,k}Q\in GT(\lambda+e_k)$, then*

$$\left(\partial_2-l_{2n-1,k}+n-1\mp\lambda_{n+1}\mp\frac{\eta_2a_1}{2a_2}\right)c(Q;a)$$
$$\pm\iota\sum_{0\le|j|\le n-1}\frac{a_{2n-2,-j}(\tau_{0,j}Q)}{l_{2n-2,j}+l_{2n-1,k}}\left(\partial_1+l_{2n-2,j}+n-1\pm\frac{\eta_2a_1}{2a_2}\right)c(\tau_{0,j}Q;a)$$
$$\mp\frac{\eta_1}{a_1}\sum_{1\le|i|\le n-1}\sum_{0\le|j|\le n-1}\frac{a_{2n-3,-i}(\tau_{i,j}Q)a_{2n-2,-j}(\tau_{0,j}Q)}{(l_{2n-3,i}-l_{2n-2,j})(l_{2n-2,j}+l_{2n-1,k})}c(\tau_{i,j}Q;a)$$
$$=0.$$

(2) *If $Q\in GT(\lambda)$, $\tau_{0,j}\sigma_{2n-1,k}Q\in GT(\lambda+e_k)$ and $\tau_{0,j}Q\notin GT(\lambda)$, then*

$$\left(\partial_1-l_{2n-2,j}+n-1\pm\frac{\eta_2a_1}{2a_2}\right)c(Q;a)$$
$$+\iota\frac{\eta_1}{a_1}\sum_{1\le|i|\le n-1}\frac{a_{2n-3,-i}(\tau_{i,0}Q)}{l_{2n-3,i}+l_{2n-2,j}}c(\tau_{i,0}Q;a)$$
$$=0.$$

(3) *If $Q\in GT(\lambda)$, $\tau_{i,j}\sigma_{2n-1,k}Q\in GT(\lambda+e_k)$, $\tau_{i,j}Q\notin GT(\lambda)$ and $\tau_{i,0}Q\notin GT(\lambda)$, then*

$$c(Q;a)=0.$$

Proof. (1) Replace $\tau_{0,j}Q$ in the second sum of the right hand of (4.9) with Q and after that replace j with $-j$. Replace $\tau_{i,j}Q$ in the third sum of (4.9) with Q and after that replace (i,j) with $(-i,-j)$. Then we have the above formula. The equations in (2) and (3) are obtained in the same way. □

We investigate the conditions in Proposition 4.10. Note that, $\tilde{\lambda}$ satisfies (4.1). Note also that, if $j=0$, then no $Q\in GT(\lambda)$ satisfies the conditions in Proposition 4.10 (2), (3).

Lemma 4.11. (1) *$Q\in GT(\lambda)$ satisfies $\sigma_{2n-1,k}Q\in GT(\lambda+e_k)$ if and only if one of the following conditions holds.*

(a) *$k=1$.*
(b) *$k\in[2,n-1]$ and $q_{2n-2,k-1}>\lambda_kD$*
(c) *$k=-(\mathrm{sgn}\lambda_n)n$.*
(d) *$k\in[-n+1,-1]$ and $q_{2n-2,-k}<\lambda_{-k}D$*
(e) *$k=(\mathrm{sgn}\lambda_n)n$ and $q_{2n-2,n-1}>|\lambda_n|D$*

(2) $Q \in GT(\lambda)$ *satisfies* $\tau_{0,j}\sigma_{2n-1,k}Q \in GT(\lambda+e_k)$ *and* $\tau_{0,j}Q \notin GT(\lambda)$ *if and only if one of the following conditions holds.*
   (a) $k \in [1, n-1]$, $j = k$, $q_{2n-2,k} = \lambda_k$ *and* $q_{2n-3,k-1} > \lambda_k D$ *The last condition is unnecessary if* $k = 1$.
   (b) $k \in [-n+1, -2]$, $j = -(-k-1) = k+1$, $q_{2n-2,-k-1} = \lambda_{-k}$ *and* $q_{2n-3,-k-1} < \lambda_{-k}D$
   (c) $k = -(\mathrm{sgn}\lambda_n)n$, $j = -(n-1)$, $q_{2n-2,n-1} = |\lambda_n|$ *and* $|q_{2n-3,n-1}| < |\lambda_n|D$

(3) $Q \in GT(\lambda)$ *satisfies* $\tau_{i,j}\sigma_{2n-1,k}Q \in GT(\lambda+e_k)$, $\tau_{i,j}Q \notin GT(\lambda)$ *and* $\tau_{i,0}Q \notin GT(\lambda)$ *if and only if one of the following conditions holds.*
   (a) $k \in [1, n-1]$, $i = j = k$, $q_{2n-3,k} = q_{2n-2,k} = \lambda_k$ *and* $q_{2n-4,k-1} > \lambda_k D$ *The last condition is unnecessary if* $k = 1$.
   (b) $k \in [-n+1, -3]$, $i = -(-k-2) = k+2$, $j = -(-k-1) = k+1$, $q_{2n-3,-k-2} = q_{2n-2,-k-1} = \lambda_{-k}$ *and* $q_{2n-4,-k-2} < \lambda_{-k}D$
   (c) $k = -(\mathrm{sgn}\lambda_n)n$, $i = -(n-2)$, $j = -(n-1)$, $q_{2n-3,n-2} = q_{2n-2,n-1} = |\lambda_n|$ *and* $q_{2n-4,n-2} < |\lambda_n|D$

We write down the differential-difference equation $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$ in the case $\Lambda \in \Xi_{m,\pm}$. In order to write equations briefly, we introduce some notation.

Notation 4.12.

$$\mathcal{D}_1^{\pm}(Q) := \partial_1 + n - 1 \pm \frac{\eta_2 a_1}{2a_2},$$

$$V_j^{\pm}(Q;a) := (\mathcal{D}_1^{\pm}(Q) + l_{2n-2,j})c(\tau_{0,j}Q;a) + \iota\frac{\eta_1}{a_1}\sum_{1\le|i|\le n-1}\frac{a_{2n-3,-i}(\tau_{i,j}Q)}{l_{2n-3,i} - l_{2n-2,j}}c(\tau_{i,j}Q;a),$$

$$\mathcal{D}_2^{\pm}(Q) := \partial_2 + \sum_{p=1}^{m-1} l_{2n-1,p} - \sum_{p=1}^{m-2} l_{2n-2,p} + m - 2 \pm \lambda_{n+1} \pm \frac{\eta_2 a_1}{2a_2},$$

$$A_j(Q) := a_{2n-2,-j}(\tau_{0,j}Q)\frac{\prod_{p=1}^{m-2}(l_{2n-2,j} - l_{2n-2,p})}{\prod_{p=1}^{m-1}(l_{2n-2,j} + l_{2n-1,-p})}$$

$$B_{j'}(j,Q) := \frac{a_{2n-2,-j'}(\tau_{0,j'}Q)\prod_{p\in[-n+1,-m+1]\cup[0,n-1]\setminus\{j\}}(l_{2n-2,j'} - l_{2n-2,p})}{\prod_{p\in[-n,-1]\cup[m,n]}(l_{2n-2,j'} + l_{2n-1,p})}.$$

Let $\Lambda \in \Xi_{m,\pm}$, $m = 2, \dots, n$. Then ϕ satisfies

$$\begin{aligned} &\mathrm{pr}_{-k,+}(\nabla_{\tilde{\lambda},\eta}\phi) = 0, && \mathrm{pr}_{-k,-}(\nabla_{\tilde{\lambda},\eta}\phi) = 0 && \text{for} \quad 1 \le k \le m-1,\\ &\mathrm{pr}_{k,\mp}(\nabla_{\tilde{\lambda},\eta}\phi) = 0, && \mathrm{pr}_{-k,\mp}(\nabla_{\tilde{\lambda},\eta}\phi) = 0 && \text{for} \quad m \le k \le n. \end{aligned}$$

By Proposition 4.10 and Lemma 4.11, we have the following lemma.

Lemma 4.13. *Suppose* $\Lambda \in \Xi_{m,\pm}$, $m \in [2, n]$. *Then* $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$ *is equivalent to the followings.*

(1) *For* $k \in [1, m-1]$, *if* $Q \in GT(\lambda)$ *satisfies* $q_{2n-2,k} < \lambda_k$, *then*

(4.10)
$$(\partial_2 - l_{2n-1,-k} + n - 1)c(Q;a) + \iota\frac{\eta_2 a_1}{2a_2}\sum_{0\le|j|\le n-1}\frac{a_{2n-2,-j}(\tau_{0,j}Q)}{l_{2n-2,j} + l_{2n-1,-k}}c(\tau_{0,j}Q;a) = 0.$$

(2) *For $k \in [-n,-1] \cup [m,n]$, if $Q \in GT(\lambda)$ satisfies one of the condition Lemma 4.11(1), then*

$$\left(\partial_2 - l_{2n-1,k} + n - 1 \pm \lambda_{n+1} \pm \frac{\eta_2 a_1}{2a_2}\right) c(Q;a) \mp \iota \sum_{0\le|j|\le n-1} \frac{a_{2n-2,-j}(\tau_{0,j}Q)}{l_{2n-2,j}+l_{2n-1,k}} V_j^{\mp}(Q;a) = 0. \tag{4.11}$$

(3) *For $k \in [2,m-1]$, if $Q \in GT(\lambda)$ satisfies $q_{2n-2,k-1} = \lambda_k$ and $q_{2n-3,k-1} < \lambda_k$, then*

$$c(Q;a) = 0, \qquad \sum_{1\le|i|\le n-1} \frac{a_{2n-3,-i}(\tau_{i,0}Q)}{l_{2n-3,i}+l_{2n-2,k-1}} c(\tau_{i,0}Q;a) = 0. \tag{4.12}$$

(4) *Suppose $Q \in GT(\lambda)$ satisfies the condition in Lemma 4.11(2) for $k \in [-n+1,-2] \cup [m,n-1] \cup \{-(\mathrm{sgn}\lambda_{n+1})n\}$. Define $j(k)$ by*

$$j(k) = \begin{cases} k & \text{if } k \in [m,n-1], \\ k+1 & \text{if } k \in [-n+1,-2], \\ -n+1 & \text{if } k = -(\mathrm{sgn}\lambda_n)n. \end{cases}$$

Then,

$$(\mathcal{D}_1^{\mp}(Q) - l_{2n-2,j(k)})c(Q;a) + \iota\frac{\eta_1}{a_1} \sum_{1\le|i|\le n-1} \frac{a_{2n-3,-i}(\tau_{i,0}Q)}{l_{2n-3,i}+l_{2n-2,j(k)}} c(\tau_{i,0}Q;a) = 0. \tag{4.13}$$

(5) *Suppose $k \in [-n+1,-3] \cup [m,n-1] \cup \{-(\mathrm{sgn}\lambda_{n+1})n\}$. If $Q \in GT(\lambda)$ satisfies the condition in Lemma 4.11(3), i.e. if*

$$\begin{aligned} &q_{2n-3,k} = q_{2n-2,k} = \lambda_k, q_{2n-4,k-1} > \lambda_k, && \text{when } k \in [m,n-1], \\ &q_{2n-3,-k-2} = q_{2n-2,-k-1} = \lambda_{-k}, q_{2n-4,-k-2} < \lambda_{-k}, && \text{when } k \in [-n+1,-3], \\ &q_{2n-3,n-2} = q_{2n-2,n-1} = |\lambda_n|, q_{2n-4,n-2} < |\lambda_n|, && \text{when } k = -(\mathrm{sgn}\lambda_n)n, \end{aligned}$$

then

$$c(Q;a) = 0.$$

By eliminating some terms of the equations in Lemma 4.13, we obtain the following equations.

Corollary 4.14. (1) *Suppose $j \in [1,m-1]$ satisfies $\tau_{0,j}Q \in GT(\lambda)$. Then*

$$\begin{aligned} &\left(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,m-1} + l_{2n-2,j} \mp \Lambda_{n+1} \mp \frac{\eta_2 a_1}{2a_2}\right) c(Q;a) \\ &+ \iota\frac{\eta_2 a_1}{2a_2} \sum_{\substack{j' \in \{j\}\cup[m,n-1] \\ \cup[-n+1,0]}} \frac{a_{2n-2,-j'}(\tau_{0,j'}Q) \prod_{1\le p\le m-1, p\ne j}(l_{2n-2,j'} - l_{2n-2,p})}{\prod_{p=1}^{m-1}(l_{2n-2,j'} + l_{2n-1,-p})} \\ &\qquad \times c(\tau_{0,j'}Q;a) \\ &= 0. \end{aligned} \tag{4.14}$$

Here, $\Lambda_{n+1} = \lambda_{n+1} \mp (n-m+1)$ is the $(n+1)$-st part of the Harish-Chandra parameter $\Lambda \in \Xi_{m,\pm}$.

(2) *Suppose* $j \in [-n+1, -m+1] \cup [0, n-1]$. *If there exists* $i \in [-n+1, n-1]$ *such that* $\tau_{i,j} Q \in GT(\lambda)$*, then*

$$(\mathcal{D}_2^{\pm}(Q) + l_{2n-2,j}) c(Q; a) \mp \iota \sum_{j' \in \{j\} \cup [-m+2,-1]} B_{j'}(j, Q) V_{j'}^{\mp}(Q; a) = 0. \tag{4.15}$$

(3) *Especially, when* $j = 0$*Cthe equation* (4.15) *is*

$$\begin{aligned} &\mathcal{D}_2^{\pm}(Q) c(Q; a) \mp \iota \sum_{j'=-m+2}^{-1} B_{j'}(0, Q) V_{j'}^{\mp}(Q; a) \\ &\mp \frac{\prod_{p=1}^{n-1} l_{2n-3,p}}{\prod_{p=m}^{n} l_{2n-1,p} \prod_{p=1}^{m-2} (-l_{2n-2,-p})} \\ &\quad \times \left(\mathcal{D}_1^{\mp}(Q) c(Q; a) + \iota \frac{\eta_1}{a_1} \sum_{1 \le |i| \le n-1} \frac{a_{2n-3,-i}(\tau_{i,0} Q)}{l_{2n-3,i}} c(\tau_{i,0} Q; a) \right) \\ &= 0 \end{aligned} \tag{4.16}$$

Proof. (4.14) can be obtained from the equations (4.10), applied to $k \in [1, m-1]$, by eliminating $c(\tau_{0,j'} Q; a)$, $j' \in [1, m-1] \setminus \{j\}$. (4.15) can be obtained from the equations (4.11), applied to $k \in [-n, -1] \cup [m, n]$, by eliminating $V_{j'}(Q; a)$, $j' \in [-n+1, -m+1] \cup [0, n-1] \setminus \{j\}$. □

Suppose $Q \in GT(\lambda)$ and $\tau_{0,-j} Q \in GT(\lambda)$ for some $j \in [-n+1, -m+1] \cup [0, n-1]$. Replace Q in (4.15) by $\tau_{0,-j} Q$. Then we obtain

$$\begin{aligned} &\mp \frac{\iota}{B_j(j, \tau_{0,-j} Q)} (\mathcal{D}_2^{\pm}(Q) - l_{2n-2,-j}) c(\tau_{0,-j} Q; a) \\ &+ \sum_{j' \in [-m+2,-1]} \frac{B_{j'}(j, \tau_{0,-j} Q)}{B_j(j, \tau_{0,-j} Q)} V_{j'}^{\mp}(\tau_{0,-j} Q; a) \\ &+ (\mathcal{D}_1^{\mp}(Q) - l_{2n-2,-j}) c(Q; a) + \iota \frac{\eta_1}{a_1} \sum_{1 \le |i| \le n-1} \frac{a_{2n-3,-i}(\tau_{i,0} Q)}{l_{2n-3,i} + l_{2n-2,-j}} c(\tau_{i,0} Q; a) \\ &= 0. \end{aligned} \tag{4.17}$$

If $Q \in GT(\lambda)$ satisfies

(1) $\tau_{0,-j} Q \notin GT(\lambda)$ and $\tau_{0,-j} \sigma_{2n-1,-j} Q \in GT(\lambda + e_{-j})$ for some $j \in [-n+1, -m]$; or
(2) $\tau_{0,-j} Q \notin GT(\lambda)$ and $\tau_{0,-j} \sigma_{2n-1,-j-1} Q \in GT(\lambda - e_{j+1})$ for some $j \in [2, n-2]$; or
(3) $\tau_{0,-n+1} Q \notin GT(\lambda)$ and $\tau_{0,-n+1} \sigma_{2n-1,-(\mathrm{sgn}\lambda_n)n} Q \in GT(\lambda - (\mathrm{sgn}\lambda_n) e_n)$,

then (4.13) holds for k such that $j(k) = -j$ (or $j(k) = n-1$ in the case (3)). We may, and do, regard this equation (4.13) as a special case of (4.17), whose first line is zero.

Lemma 4.15. *If* (4.17) *holds for* $j = j_1, \dots, j_s$ *and* $\tau_{i,0}Q \in GT(\lambda)$ *for* $i = i_1, \dots, i_{s-1}$, *then*

$$
\begin{aligned}
(4.18)\qquad & \pm \sum_{\mu=1}^{s} \frac{\iota}{B_{j_\mu}(j_\mu, \tau_{0,-j_\mu}Q)} \frac{\prod_{\nu=1}^{s-1}(l_{2n-2,-j_\mu} + l_{2n-3,i_\nu})}{\prod_{\nu=1,\neq\mu}^{s}(l_{2n-2,-j_\mu} - l_{2n-2,-j_\nu})} \\
&\qquad\qquad \times (\mathcal{D}_2^{\pm}(Q) - l_{2n-2,-j_\mu})c(\tau_{0,-j_\mu}Q; a) \\
& + \sum_{\mu=1}^{s} \sum_{j' \in [-m+2,-1]} \frac{\prod_{\nu=1}^{s-1}(l_{2n-2,-j_\mu} + l_{2n-3,i_\nu})}{\prod_{\nu=1,\neq\mu}^{s}(l_{2n-2,-j_\mu} - l_{2n-2,-j_\nu})} \frac{B_{j'}(j_\mu, \tau_{0,-j_\mu}Q)}{B_{j_\mu}(j_\mu, \tau_{0,-j_\mu}Q)} \\
&\qquad\qquad \times V_{j'}^{\mp}(\tau_{0,-j_\mu}Q; a) \\
& + \left(\mathcal{D}_1^{\mp}(Q) - \sum_{\mu=1}^{s} l_{2n-2,-j_\mu} - \sum_{\nu=1}^{s-1} l_{2n-3,i_\nu}\right) c(Q; a) \\
& + \iota \frac{\eta_1}{a_1} \sum_{1 \le |i| \le n-1} \frac{a_{2n-3,-i}(\tau_{i,0}Q) \prod_{\nu=1}^{s-1}(l_{2n-3,i} - l_{2n-3,i_\nu})}{\prod_{\mu=1}^{s}(l_{2n-3,i} + l_{2n-2,-j_\mu})} c(\tau_{i,0}Q; a) \\
& = 0.
\end{aligned}
$$

Proof. This is obtained from the equations (4.17), applied to $j = j_1, \dots, j_s$, by eliminating $c(\tau_{i,0}Q; a)$, $i = i_1, \dots, i_{s-1}$. $\square$

We apply this lemma to some special cases. For $m' \in [m-1, n-1]$ and $Q \in GT(\lambda)$, let

$$
\begin{aligned}
K_1(m') &:= \{p \in [1, m'-1] | \tau_{p,0}Q \in GT(\lambda)\}, \\
K_2(m') &:= [1, m'-1] \setminus K_1(m'), \\
K_3(m') &:= \{p \in [-n+2, -m'] | \tau_{p,0}Q \in GT(\lambda)\}, \\
K_4(m') &:= [-n+2, -m'] \setminus K_3(m'), \\
K_5(m') &:= \{p - 1 | p \in K_3(m')\}.
\end{aligned}
$$

If $p \in K_1(m')$, then $\tau_{0,-p}Q \in GT(\lambda)$ or $\lambda_{p+1} = q_{2n-2,p} > q_{2n-3,p} \ge q_{2n-2,p+1}$. In the latter case, $\tau_{0,-p}\sigma_{2n-1,-p-1}Q \in GT(\lambda - e_{p+1})$. Thus (4.17) holds for $j = p$ in each case. If $p \in K_3(m')$, then $\tau_{0,-p+1}Q \in GT(\lambda)$ or $q_{2n-2,-p} \ge q_{2n-3,-p} > q_{2n-2,-p+1} = \lambda_{-p+1}$. In the latter case, $\tau_{0,-p+1}\sigma_{2n-1,-p+1}Q \in GT(\lambda + e_{-p+1})$. Thus (4.17) holds for $j = p-1$ in each case.

Let $i \in K_1(m') \cup K_3(m')$, or $i = -n+1$ if $\tau_{-n+1,0}Q \in GT(\lambda)$. Define

$$
\begin{aligned}
I_1(m') &:= K_1(m') \cup K_5(m') \cup \{0\}, \\
I_2(m') &:= \begin{cases} K_1(m') \cup K_3(m') \cup \{-n+1\} & \text{if } \tau_{-n+1,0}Q \in GT(\lambda) \\ K_1(m') \cup K_3(m') \cup \{n-1\} & \text{if } \tau_{-n+1,0}Q \notin GT(\lambda), \end{cases} \\
I_2(m', i) &:= I_2(m') \setminus \{i\}.
\end{aligned}
$$

Then (4.18) holds for $\{j_1, \dots, j_s\} = I_1(m')$ and $\{i_1, \dots, i_{s-1}\} = I_2(m', i)$, so the following formula is obtained.

Corollary 4.16. *For $i \in K_1(m') \cup K_3(m')$ or $i = -n+1$ if $\tau_{-n+1,0}Q \in GT(\lambda)$, then*

(4.19)

$$\begin{aligned}
&-\iota\frac{\eta_1}{a_1}\frac{a_{2n-3,-i}(\tau_{i,0}Q)\prod_{i'\in I_2(m',i)}(l_{2n-3,i}-l_{2n-3,i'})}{\prod_{j\in I_1(m')}(l_{2n-3,i}+l_{2n-2,-j})}c(\tau_{i,0}Q;a)\\
&=\pm\sum_{j\in I_1(m')}\frac{\iota}{B_j(j,\tau_{0,-j}Q)}\frac{\prod_{i'\in I_2(m',i)}(l_{2n-2,-j}+l_{2n-3,i'})}{\prod_{j'\in I_1(m')\setminus\{j\}}(l_{2n-2,-j}-l_{2n-2,-j'})}\\
&\qquad\qquad\times(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,-j})c(\tau_{0,-j}Q;a)\\
&\quad+\sum_{j\in I_1(m')}\sum_{j'\in[-m+2,-1]}\frac{\prod_{i'\in I_2(m',i)}(l_{2n-2,-j}+l_{2n-3,i'})}{\prod_{j''\in I_1(m')\setminus\{j\}}(l_{2n-2,-j}-l_{2n-2,-j''})}\frac{B_{j'}(j,\tau_{0,-j}Q)}{B_j(j,\tau_{0,-j}Q)}\\
&\qquad\qquad\times V_{j'}^{\mp}(\tau_{0,-j}Q;a)\\
&\quad+\left(\mathcal{D}_1^{\mp}(Q)-\sum_{j\in I_1(m')}l_{2n-2,-j}-\sum_{i'\in I_2(m',i)}l_{2n-3,i'}\right)c(Q;a)\\
&\quad+\iota\frac{\eta_1}{a_1}\sum_{\substack{i'\in[-m'+1,-1]\\ \cup[m',n-1]}}\frac{a_{2n-3,-i'}(\tau_{i',0}Q)\prod_{i''\in I_2(m',i)}(l_{2n-3,i'}-l_{2n-3,i''})}{\prod_{j\in I_1(m')}(l_{2n-3,i'}+l_{2n-2,-j})}c(\tau_{i',0}Q;a).
\end{aligned}$$

Suppose $Q \in GT(\lambda)$ satisfies

$$q_{2n-2,p} = q_{2n-3,p} = q_{2n-4,p} \quad \text{for any } p \in [1, m-2]. \tag{4.20}$$

Define

$$K_6(m') := \{j \in [m-1, m'] \cup [-n+1, -m'] | \tau_{0,j}Q \in GT(\lambda)\}. \tag{4.21}$$

For $j \in K_6(m')$, define

$$I_1(m', j) := I_1(m') \cup \{-j\}.$$

Then (4.18) holds for $\{j_1, \dots, j_s\} = I_1(m', j)$ and $\{i_1, \dots, i_{s-1}\} = I_2(m')$. By (4.20) and the definition of $V_j^{\pm}(Q;a)$, the term $V_{j''}(\tau_{0,-j'}Q)$ is zero for $j' \in [-m+2, -1]$ and $j'' \in I_1(m', j)$. Thus we obtain the following formula.

Corollary 4.17. *Suppose $Q \in GT(\lambda)$ satisfies (4.20). If $j \in K_6(m')$, then*

(4.22)

$$\begin{aligned}
&\mp\frac{\iota}{B_{-j}(-j,\tau_{0,j}Q)}\frac{\prod_{i\in I_2(m')}(l_{2n-2,j}+l_{2n-3,i})}{\prod_{j'\in I_1(m')}(l_{2n-2,j}-l_{2n-2,-j'})}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,j})c(\tau_{0,j}Q;a)\\
&=\pm\sum_{j'\in I_1(m')}\frac{\iota}{B_{j'}(j',\tau_{0,-j'}Q)}\frac{\prod_{i\in I_2(m')}(l_{2n-2,-j'}+l_{2n-3,i})}{\prod_{j''\in I_1(m',j)\setminus\{j'\}}(l_{2n-2,-j'}-l_{2n-2,-j''})}\\
&\qquad\times(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,-j'})c(\tau_{0,-j'}Q;a)\\
&\quad+\left(\mathcal{D}_1^{\mp}(Q)-\sum_{j'\in I_1(m',j)}l_{2n-2,-j'}-\sum_{i\in I_2(m')}l_{2n-3,i}\right)c(Q;a)\\
&\quad+\iota\frac{\eta_1}{a_1}\sum_{\substack{i\in[-m'+1,-m+1]\\ \cup[m',n-1]}}\frac{a_{2n-3,-i}(\tau_{i,0}Q)\prod_{i'\in I_2(m')}(l_{2n-3,i}-l_{2n-3,i'})}{\prod_{j'\in I_1(m',j)}(l_{2n-3,i}+l_{2n-2,-j'})}c(\tau_{i,0}Q;a).
\end{aligned}$$

4.3. **Condition for $\mathbf{q}_{2n-4}$.** In this subsection, we deduce necessary conditions for $\mathbf{q}_{2n-4}$ so that the differential-difference equation has a non-trivial solution. In the first place, we deduce conditions for $q_{2n-4,k}$, $k \in [1, m-2]$.

Lemma 4.18. *Suppose $k \in [1, m-2]$. If $Q \in GT(\lambda)$ satisfies $q_{2n-4,k} < \lambda_{k+1}$, then $c(Q;a) = 0$.*

Proof. **Step 1.** Among the Gelfand-Tsetlin patterns with $q_{2n-4,k} < \lambda_{k+1}$, there is a Q satisfying $q_{2n-2,k} = \lambda_{k+1}$ and $q_{2n-3,k} < \lambda_{k+1}$. By the first equation in (4.12), $c(Q;a) = 0$ for this Q. Therefore, by the second equation in (4.12), $c(Q;a) = 0$ for $Q \in GT(\lambda)$ satisfying $q_{2n-2,k} = q_{2n-3,k} = \lambda_{k+1}$. Thus, $c(Q;a) = 0$ for all $Q \in GT(\lambda)$ such that $q_{2n-2,k} = \lambda_{k+1}$ and $q_{2n-3,k} \leq \lambda_{k+1}$.
Step 2. Suppose $Q \in GT(\lambda)$ satisfies $q_{2n-4,k} < \lambda_{k+1}$ and $q_{2n-4,k} \leq q_{2n-3,k} \leq q_{2n-2,k} = \lambda_{k+1}$. In this case, if $j' \in [m, n-1] \cup [-n+1, 0] \setminus \{k\}$, then $c(\tau_{0,j'}Q;a) = 0$ by Step 1, since $\tau_{0,j'}Q$ satisfies the condition in it. Then by (4.14), applied to this Q and $j = k$, we know that $c(Q;a) = 0$ for $Q \in GT(\lambda)$ satisfying $q_{2n-2,k} = \lambda_{k+1} + 1$ and $q_{2n-3,k} \leq \lambda_{k+1}$. By repeating this discussion, we know that $c(Q;a) = 0$ for $Q \in GT(\lambda)$ satisfying $\lambda_k \geq q_{2n-2,k} \geq \lambda_{k+1}$ and $q_{2n-3,k} \leq \lambda_{k+1}$.
Step 3. Suppose $Q \in GT(\lambda)$ satisfies $q_{2n-2,k} = q_{2n-3,k} = \lambda_{k+1}$. By Step 2, $c(Q;a) = 0$. If $\tau_{i,j}Q \in GT(\lambda)$ for $(i,j) \neq (k,k)$, then it satisfies $\lambda_k \geq q_{2n-2,k} \geq \lambda_{k+1}$ and $q_{2n-3,k} \leq \lambda_{k+1}$; so it satisfies the condition in the conclusion of Step 2. It follows that $c(\tau_{i,j}Q;a) = 0$ for $(i,j) \neq (k,k)$. Then by (4.15), applied to this Q and $j = k$, we have $c(\tau_{k,k}Q;a) = 0$, that is to say $c(Q;a) = 0$ for $Q \in GT(\lambda)$ satisfying $q_{2n-4,k} < \lambda_{k+1}$ and $q_{2n-2,k} = q_{2n-3,k} = \lambda_{k+1} + 1$.
Step 4. Suppose $Q \in GT(\lambda)$ satisfies the condition in the conclusion of Step 3. Then, by applying the discussion in Step 2 for this Q, we know that $c(Q;a) = 0$ for $Q \in GT(\lambda)$ satisfying $q_{2n-3,k} \leq \lambda_{k+1} + 1$.

By repeating the shift operations as in Step 2 and Step 3, the lemma is shown. □

In the second place, we deduce conditions for $q_{2n-4,k}$, $k \in [m-1, n-2]$.

Lemma 4.19. *Suppose $k \in [m-1, n-2]$. If $Q \in GT(\lambda)$ satisfies $q_{2n-4,k} > \lambda_{k+1}$, then $c(Q;a) = 0$.*

Proof. **Step 1.** Among the Gelfand-Tsetlin patterns with $q_{2n-4,k} > \lambda_{k+1}$, there is a Q satisfying $q_{2n-3,k+1} = q_{2n-2,k+1} = \lambda_{k+1}$. By Lemma 4.13 (5), $c(Q;a) = 0$ for this Q.
Step 2. We know $c(Q;a) = 0$ for those Q with $q_{2n-3,k+1} = q_{2n-2,k+1} = \lambda_{k+1}$. Shift $q_{2n-3,k+1}$ downward by using (4.16). Then we know that $c(Q;a) = 0$ for all $Q \in GT(\lambda)$ satisfying $q_{2n-4,k} > \lambda_{k+1}$ and $\lambda_{k+1} = q_{2n-2,k+1} \geq q_{2n-3,k+1}$.
Step 3. Suppose $Q \in GT(\lambda)$ satisfies the condition in Step 1. Then by (4.15), applied to this Q and $j = -k-1$, we know that $c(Q;a) = 0$ for all $Q \in GT(\lambda)$ satisfying $q_{2n-4,k} > \lambda_{k+1}$ and $q_{2n-2,k+1} = q_{2n-3,k+1} = \lambda_{k+1} - 1$.

By repeating the shift operations as in Step 2 and Step 3, the lemma is shown. □

Lemma 4.20. *If $Q \in GT(\lambda)$ satisfies $q_{2n-4,k} < \lambda_{k+2}$ for $k \in [m-1, n-3]$, or $q_{2n-4,n-2} < |\lambda_n|$, then $c(Q;a) = 0$.*

Proof. The proof of this lemma is just the same as that of the previous one. In the first place, if Q satisfies the above condition and $q_{2n-3,k} = q_{2n-2,k+1} = \lambda_{k+2}$ (or $= |\lambda_n|$ if $k = n-2$), we have $c(Q;a) = 0$ by Lemma 4.13 (5). Shift $q_{2n-3,k}$ upward by using (4.16). Then we know that $c(Q;a) = 0$ for all $Q \in GT(\lambda)$ satisfying $q_{2n-4,k} <$

λ_{k+2} and $q_{2n-2,k+1} = \lambda_{k+2}$. In the third place, if $Q \in GT(\lambda)$ satisfies $q_{2n-3,k} = q_{2n-2,k+1} = \lambda_{k+2}$, then shift upward them to $q_{2n-3,k} = q_{2n-2,k+1} = \lambda_{k+2} + 1$ by using (4.15). By repeating these shift operations, the lemma is shown. □

Corollary 4.21. *If $Q \in GT(\lambda)$ does not satisfy*

$$\begin{cases} \lambda_k \geq q_{2n-4,k} \geq \lambda_{k+1} & \text{for } k \in [1, m-2], \\ \lambda_{k+1} \geq q_{2n-4,k} \geq \lambda_{k+2} & \text{for } k \in [m-1, n-3], \\ \lambda_{n-1} \geq q_{2n-4,n-2} \geq |\lambda_n|, \end{cases} \tag{4.23}$$

then $c(Q; a) = 0$.

4.4. **Reduction to the "corner" vectors.** Choose a $\mathbf{q}_{2n-4}$ satisfying the condition (4.23). In this subsection, we show that, if $c(Q_0; a)$ is known for some $Q_0 \in GT(\lambda)$ containing this $\mathbf{q}_{2n-4}$, then it completely determines the other $c(Q; a)$'s for $Q \in GT(\lambda)$ containing the same $\mathbf{q}_1, \ldots, \mathbf{q}_{2n-4}$ parts. We call Q_0 with this property a "corner vector".

As will be seen in §5, there is a set of Gelfand-Tsetlin bases Q, including corner vectors, such that we can explicitly write down the scalar differential equations satisfied by $c(Q; a)$. Such bases $Q \in GT(\lambda)$ satisfy the following conditions: For an $m' \in [m-1, n-1]$,

$$\begin{cases} q_{2n-2,p} = \lambda_{p+1}, \ \ q_{2n-3,p} = q_{2n-4,p} & \text{for } p \in [m-1, m'-1], \\ q_{2n-2,p} = \lambda_p, \ \ q_{2n-3,p} = q_{2n-4,p-1} & \text{for } p \in [m'+1, n-1], \\ q_{2n-2,m'} = q_{2n-3,m'} \in [\lambda_{m'+1}, q_{2n-4,m'-1}] & \text{if } m' \geq m, \\ \qquad\qquad\qquad\qquad \text{or } \ [\lambda_m, \lambda_{m-1}] & \text{if } m' = m-1. \end{cases} \tag{4.24}$$

Definition 4.22. $Q^-_{m'} \in GT(\lambda)$ is the Gelfand-Tsetlin pattern which satisfies (4.20), (4.23), (4.24) and $q_{2n-2,m'} = q_{2n-3,m'} = \lambda_{m'+1}$. Analogously, $Q^+_{m'} \in GT(\lambda)$ is the Gelfand-Tsetlin pattern which satisfies (4.20), (4.23), (4.24) and $q_{2n-2,m'} = q_{2n-3,m'} = q_{2n-4,m'-1}$ (or $= \lambda_{m-1}$ if $m' = m-1$).

Theorem 4.23. *If $c(Q^+_{n-1}; a)$ is known, then it determines all the $c(Q; a)$ for $Q \in GT(\lambda)$ containing the same $\mathbf{q}_1, \ldots, \mathbf{q}_{2n-4}$ parts as Q^+_{n-1}.*

Proof. In this proof, we assume all $Q \in GT(\lambda)$ contain the same $\mathbf{q}_1, \ldots, \mathbf{q}_{2n-4}$ as Q^+_{n-1}. Note that this $\mathbf{q}_{2n-4}$ satisfies (4.23). For simplicity, we prove the case when $\lambda_n > 0$ and $m < n$. The case when $\lambda_n < 0$ is proved just in the same way. The case when $m = n$ is also proved just in the same way, and the proof is a little easier.

For $Q = (\mathbf{q}_1, \ldots, \mathbf{q}_{2n-1}) \in GT(\lambda)$, define lengths of Q by

$$\|Q\|_{2n-2,1} := \sum_{j=1}^{m-2} (q_{2n-2,j} - \lambda_{j+1}), \qquad \|Q\|_{2n-2,2} := \sum_{j=m-1}^{n-2} (q_{2n-2,j} - \lambda_{j+1}),$$

$$\|Q\|_{2n-3,1} := \sum_{j=1}^{m-2} (q_{2n-3,j} - q_{2n-4,j}),$$

$$\|Q\|_{2n-3,2} := \sum_{j=m-1}^{n-3} (q_{2n-3,j} - q_{2n-4,j}) + q_{2n-3,n-2} - q_{2n-3,n-1} - |q_{2n-2,n-1} - q_{2n-4,n-2}|.$$

Note that, since $q_{2n-3,n-2} \geq q_{2n-2,n-1} \geq q_{2n-3,n-1}$ and $q_{2n-3,n-2} \geq q_{2n-4,n-2} \geq q_{2n-3,n-1}$, the term

$$\begin{aligned}
&q_{2n-3,n-2} - q_{2n-3,n-1} - |q_{2n-2,n-1} - q_{2n-4,n-2}| \\
&\quad = q_{2n-3,n-2} - q_{2n-3,n-1} \\
&\qquad - \max\{q_{2n-2,n-1}, q_{2n-4,n-2}\} + \min\{q_{2n-2,n-1}, q_{2n-4,n-2}\}
\end{aligned}$$

is zero if and only if either (i) $q_{2n-3,n-2} = q_{2n-2,n-1}$ and $q_{2n-3,n-1} = q_{2n-4,n-2}$ or (ii) $q_{2n-3,n-2} = q_{2n-4,n-2}$ $q_{2n-3,n-1} = q_{2n-2,n-1}$. Define a partial order $Q' \prec Q$ in $GT(\lambda)$ by

$$\begin{aligned}
Q' \preceq Q \quad &\Leftrightarrow \quad Q' = Q \text{ or } Q' \prec Q, \\
Q' \prec Q \quad &\Leftrightarrow \quad \|Q'\|_{2n-2,1} < \|Q\|_{2n-2,1}; \\
&\qquad \text{or } \|Q'\|_{2n-2,1} = \|Q\|_{2n-2,1} \text{ and } \|Q'\|_{2n-3,1} < \|Q\|_{2n-3,1}; \\
&\qquad \text{or } \|Q'\|_{2n-2,1} = \|Q\|_{2n-2,1}, \|Q'\|_{2n-3,1} = \|Q\|_{2n-3,1} \\
&\qquad\qquad \text{and } \|Q'\|_{2n-2,2} < \|Q\|_{2n-2,2}; \\
&\qquad \text{or } \|Q'\|_{2n-2,1} = \|Q\|_{2n-2,1}, \|Q'\|_{2n-3,1} = \|Q\|_{2n-3,1}, \\
&\qquad\quad \|Q'\|_{2n-2,2} = \|Q\|_{2n-2,2} \text{ and } \|Q'\|_{2n-3,2} < \|Q\|_{2n-3,2}.
\end{aligned}$$

We will show Theorem 4.23 by induction on this partial order.
Step 1. If Q satisfies (4.20) and

$$\begin{cases} q_{2n-2,p} = \lambda_{p+1} & \text{for } p \in [m-1, n-2], \\ q_{2n-3,p} = q_{2n-4,p} & \text{for } p \in [m-1, n-3], \\ \lambda_{n-1} > q_{2n-3,n-2} = q_{2n-2,n-1} \geq q_{2n-4,n-2}, \\ q_{2n-3,n-1} = q_{2n-4,n-2}, \end{cases} \tag{4.25}$$

then the equation (4.15), applied to this Q and $j = n-1$, is

$$\begin{aligned}
&(\mathcal{D}_2(Q) + l_{2n-2,n-1})c(Q;a) \\
&+ \frac{\eta_1}{a_1} \frac{B_{n-1}(n-1,Q)a_{2n-3,-n+2}(\tau_{n-2,n-1}Q)}{l_{2n-3,n-2} - l_{2n-2,n-1}} c(\tau_{n-2,n-1}Q;a) \\
&= 0.
\end{aligned}$$

If Q satisfies (4.20) and

$$\begin{cases} q_{2n-2,p} = \lambda_{p+1}, & \text{for } p \in [m-1, n-2], \\ q_{2n-3,p} = q_{2n-4,p} & \text{for } p \in [m-1, n-2], \\ q_{2n-4,n-2} \geq q_{2n-2,n-1} = q_{2n-3,n-1} > \lambda_n, \end{cases} \tag{4.26}$$

then the equation (4.15) for this Q and $j = -(n-1)$ is

$$\begin{aligned}
&(\mathcal{D}_2(Q) + l_{2n-2,-n+1})c(Q;a) \\
&+ \frac{\eta_1}{a_1} \frac{B_{-n+1}(-n+1,Q)a_{2n-3,n-1}(\tau_{-n+1,-n+1}Q)}{l_{2n-3,-n+1} - l_{2n-2,-n+1}} c(\tau_{-n+1,-n+1}Q;a) \\
&= 0.
\end{aligned}$$

By repeating these shift operations, we know that $c(Q^+_{n-1};a)$ determines the $c(Q;a)$ for $Q \in GT(\lambda)$ which satisfies (i) (4.20) and (4.25) or (ii) (4.20) and (4.26), in other words, for Q which is minimal with respect to the partial order $\preceq$.

Step 2. Suppose $Q \in GT(\lambda)$ in question satisfies $\tau_{i,0}Q \in GT(\lambda)$ for an $i \in [1, n-2] \cup \{-n+1\}$. Then $\tau_{i,0}Q \succ Q$ since $\|\tau_{i,0}Q\|_{2n-2,p} = \|Q\|_{2n-2,p}$ for $p = 1, 2$ but $\|\tau_{i,0}Q\|_{2n-3,p} > \|Q\|_{2n-3,p}$ for $p = 1$ or 2. Consider the equation (4.19) applied to $m' = n-1$. If $j \in I_1(n-1) \setminus \{0\} = K_1(n-1) \subset [1, n-2]$, then $\tau_{0,-j}Q \prec Q$ since $\|\tau_{0,-j}Q\|_{2n-2,p} < \|Q\|_{2n-2,p}$ for $p = 1$ or 2. If $j \in I_1(n-1)$, $j' \in [-m+2, -1]$ and $i' \in [-n+1, n-1]$, then $\tau_{i',j'}\tau_{0,j}Q \prec Q$ since $\|\tau_{i',j'}\tau_{0,j}Q\|_{2n-2,1} < \|Q\|_{2n-2,1}$. Moreover, if $i' \in [-n+2, -1] \cup \{n-1\}$, then $\tau_{i',0}Q \prec Q$ since $\|\tau_{i',0}Q\|_{2n-3,p} < \|Q\|_{2n-3,p}$ for $p = 1$ or 2. It follows that all the $c(Q'; a)$'s appearing in the right hand of (4.19) satisfy $Q' \preceq Q$. Therefore, $c(\tau_{i,0}Q; a)$ can be expressed as

$$c(\tau_{i,0}Q; a) = \sum_{Q' \preceq Q (\prec \tau_{i,0}Q)} (\text{differential of } c(Q'; a)),$$

and it is determined by $c(Q^+_{n-1}; a)$ by the hypothesis of induction. Especially, if $Q \in GT(\lambda)$ satisfies (4.20) and $q_{2n-2,p} = \lambda_{p+1}$ for any $p \in [m-1, n-2]$, then we know from the result of Step 1 that $c(Q^+_{n-1}; a)$ determines $c(Q; a)$.

Step 3. Suppose $Q \in GT(\lambda)$ satisfies (4.20). Let $k \in K_6(n-1)$. Consider the equation (4.22) applied to $m' = n-1$ and $j = k$. Since $j' \in I_1(n-1)$ implies $\tau_{0,-j'}Q \preceq Q$ and $i \in [-n+2, -m+1] \cup \{n-1\}$ implies $\tau_{i,0}Q \prec Q$, all the $c(Q'; a)$ appearing in the right hand of (4.22) satisfies $Q' \preceq Q$. Therefore, $c(\tau_{0,k}Q; a)$ satisfies

$$(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,k})c(\tau_{0,k}Q; a) = \sum_{Q' \preceq Q} (\text{differential of } c(Q'; a)). \tag{4.27}$$

The equation (4.14) for $j = m-1$ implies

$$\sum_{k \in K_6(n-1)} A_k(Q)c(\tau_{0,k}Q; a) = \sum_{Q' \preceq Q} (\text{differential of } c(Q'; a)) \tag{4.28}$$

since $\tau_{0,j'}Q \prec Q$ for $j' \in [-n+2, -m+1]$.

In order to eliminate extra terms in the left hand of (4.28), consider the following differential operator. For any $m' \in [m-1, n-1]$ and $j, k \in K_6(m')$, the fraction in the right hand of

$$\begin{aligned} &\prod_{p \in K_6(m') \setminus \{j\}} (\mathcal{D}_2^{\pm}(Q) - l_{2n-2,p}) \\ &= \frac{\prod_{p \in K_6(m') \setminus \{j\}}(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,p}) - \prod_{p \in K_6(m') \setminus \{j\}}(l_{2n-2,k} - l_{2n-2,p})}{\mathcal{D}_2^{\pm}(Q) - l_{2n-2,k}} \\ &\quad \times (\mathcal{D}_2^{\pm}(Q) - l_{2n-2,k}) \\ &\quad + \prod_{p \in K_6(m') \setminus \{j\}} (l_{2n-2,k} - l_{2n-2,p}) \end{aligned} \tag{4.29}$$

is a polynomial in $\mathcal{D}_2^{\pm}(Q)$, so it is a differential operator. Note that the last term in the right hand is zero if $k \neq j$.

Choose $j \in K_6(n-1) \cap [m-1, n-2]$. We have $\tau_{0,j}Q \succ Q$, since $j \in [m-1, n-2]$ implies $\|\tau_{0,j}Q\|_{2n-2,2} > \|Q\|_{2n-2,2}$. Differentiate both sides of (4.28) by $\prod_{p \in K_6(n-1) \setminus \{j\}}(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,p})$ and use (4.27) and (4.29). Then we get

$$c(\tau_{0,j}Q; a) = \sum_{Q' \prec \tau_{0,j}Q} (\text{differential of } c(Q'; a)).$$

By the hypothesis of induction, $c(\tau_{0,j}Q;a)$ is determined by $c(Q^+_{n-1};a)$, and so is $c(\tau_{i,j}Q;a)$ for $i \in [m-1, n-2]$ because of Step 2. Then we know from the result of Step 2 that $c(Q^+_{n-1};a)$ determines the $c(Q;a)$ for all $Q \in GT(\lambda)$ satisfying (4.20).

Step 4. If $\tau_{0,j}Q \in GT(\lambda)$ for $j \in [1, m-2]$, then the equation (4.14), applied to this j, implies

$$c(\tau_{0,j}Q;a) = \sum_{Q' \prec \tau_{0,j}Q} (\text{differential of } c(Q';a)).$$

By the hypothesis of induction, $c(\tau_{0,j}Q;a)$ is determined by $c(Q^+_{n-1};a)$, and so is $c(\tau_{i,j}Q;a)$ for $i \in [1, m-1]$ because of Step 2. Then we know from the result of Step 3 that $c(Q^+_{n-1};a)$ determines all the $c(Q;a)$ for $Q \in GT(\lambda)$. $\square$

5. Determination of Whittaker models

In this section, we first deduce a system of differential equations which are satisfied by $c(Q;a)$ for $Q \in GT(\lambda)$ satisfying (4.20), (4.23) and (4.24). Secondly obtained are the Mellin-Barnes type integral formulas of the solutions of this system of equations. Though the dimension of the solution space is high, only a few of the solutions satisfy the whole differential-difference equations $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$. Lastly, continuous intertwining operators are determined.

5.1. **Scalar differential equations.** In this subsection, we assume that Q satisfies (4.20), (4.23) and (4.24). In this case,

$$\begin{aligned}
K_1(m') &= \{p \in [m-1, m'-1] | q_{2n-3,p} < q_{2n-2,p}\},\\
K_2(m') &= [1, m'-1] \setminus K_1 = [1, m-2] \cup \{p \in [m-1, m'-1] | q_{2n-3,p} = q_{2n-2,p}\},\\
K_3(m') &= \{p \in [-n+2, -m'] | q_{2n-3,-p} > q_{2n-2,-p+1} = \lambda_{-p+1}\},\\
K_4(m') &= [-n+2, -m'-1] \setminus K_3\\
&= \{p \in [-n+2, -m'-1] | q_{2n-3,-p} = q_{2n-2,-p+1} = \lambda_{-p+1}\},\\
K_5(m') &= \{p-1 | p \in K_3(m')\}\\
&= \{p \in [-n+1, -m'-1] | q_{2n-3,-p-1} > q_{2n-2,-p} = \lambda_{-p}\},
\end{aligned}$$

by definition.

Lemma 5.1. *Let*

$$\begin{aligned}
J(m') &:= [-n+1, -m'-1] \cup [m-1, m'-1],\\
d_{m'}(Q) &:= \sum_{p \in J(m') \cup \{m'\}} l_{2n-2,-p} + \sum_{p \in J(m')} l_{2n-3,p}.
\end{aligned}$$

Suppose $Q \in GT(\lambda)$ satisfies (4.20), (4.23) and (4.24).

(1) *If $\tau_{-m',0}Q \in GT(\lambda)$, then*

$$\begin{aligned}
&-\iota\frac{\eta_1}{a_1}\frac{a_{2n-3,m'}(\tau_{-m',0}Q)\prod_{p\in J(m')}(l_{2n-3,-m'} - l_{2n-3,p})}{\prod_{p \in J(m') \cup \{0\}}(l_{2n-3,-m'} + l_{2n-2,-p})} c(\tau_{-m',0}Q;a)\\
&= \left(\pm\frac{l_{2n-1,n}}{l_{2n-2,-m'}}\mathcal{D}^{\pm}_2(Q) + \mathcal{D}^{\mp}_1(Q) - d_{m'}(Q) + l_{2n-2,-m'}\right) c(Q;a).
\end{aligned} \tag{5.1}$$

(2) *If* $q_{2n-2,m'} = q_{2n-3,m'} > \lambda_{m'+1}$, *then*

(5.2)
$$
\begin{aligned}
&\mp \frac{\iota}{B_{m'}(m', \tau_{-m',-m'}Q)} \frac{(2l_{2n-2,-m'}+1)\prod_{p\in J(m')}(l_{2n-2,-m'}+l_{2n-3,p})}{\prod_{p\in J(m')\cup\{0\}}(l_{2n-2,-m'}-l_{2n-2,-p})} \\
&\qquad \times (\mathcal{D}_2^{\pm}(Q) - l_{2n-2,-m'})c(\tau_{-m',-m'}Q;a) \\
&\quad = \left(\mp \frac{l_{2n-1,n}}{l_{2n-2,-m'}}\mathcal{D}_2^{\pm}(Q) + \mathcal{D}_1^{\mp}(Q) - d_{m'}(Q) - l_{2n-2,-m'} - 1\right) c(\tau_{-m',0}Q;a) \\
&\qquad + \iota\frac{\eta_1}{a_1}\frac{a_{2n-3,-m'}(Q)(2l_{2n-3,m'}-2)\prod_{p\in J(m')}(l_{2n-3,m'}-l_{2n-3,p}-1)}{\prod_{p\in J(m')\cup\{0,m'\}}(l_{2n-3,m'}+l_{2n-2,-p}-1)}c(Q;a).
\end{aligned}
$$

(3) *If* $q_{2n-2,m'} = q_{2n-3,m'} > \lambda_{m'+1}$, *then*

$$
\begin{aligned}
(5.3)\quad c(\tau_{-m',-m'}Q;a) = &\mp\frac{a_1}{\eta_1}\frac{l_{2n-3,-m'}-l_{2n-2,-m'}+1}{a_{2n-2,m'}(\tau_{-m',-m'}Q)a_{2n-3,m'}(\tau_{-m',0}Q)} \\
&\times \frac{\prod_{p\in[-n,-1]\cup[m,n]}(l_{2n-2,-m'}+l_{2n-1,p})}{\prod_{p\in[-n+1,-m+1]\cup[0,n-1]\setminus\{-m'\}}(l_{2n-2,-m'}-l_{2n-2,p})} \\
&\times (\mathcal{D}_2^{\pm}(Q) + l_{2n-2,-m'})c(Q;a).
\end{aligned}
$$

(4) *For* $k \in K_6(m')$

(5.4)
$$
\begin{aligned}
&\mp \frac{\iota}{B_{-k}(-k,\tau_{0,k}Q)} \frac{\prod_{p\in J(m')\cup\{-m'\}}(l_{2n-2,k}+l_{2n-3,p})}{\prod_{p\in J(m')\cup\{0\}}(l_{2n-2,k}-l_{2n-2,-p})}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,k})c(\tau_{0,k}Q;a) \\
&\quad = \left(\mp\frac{l_{2n-1,n}}{l_{2n-2,k}}\mathcal{D}_2^{\pm}(Q) + \mathcal{D}_1^{\mp}(Q) - d_{m'}(Q) - l_{2n-2,k}\right)c(Q;a)
\end{aligned}
$$

Proof. (1) Since $-m' \in K_3(m')$, the equation (4.19) holds for $i = -m'$. If $j \in I_1(m') \setminus \{0\} = K_1(m') \cup K_5(m')$, then $\tau_{0,-j}Q \notin GT(\lambda)$ because of (4.24). If $j' \in [-m+2,-1]$ and $j \in I_1(m')$, then $\tau_{i',j'}\tau_{0,-j}Q \notin GT(\lambda)$ for all i' because of (4.20). If $i' \in [-m'+1,-1] \cup [m', n-1]$, then $\tau_{i',0}Q \notin GT(\lambda)$ because of (4.20) and (4.24). Therefore, the terms in the right hand of (4.19) vanish except for the third line and the $(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,0})c(Q;a)$ term in the first line. If we arrange the coefficients by using $l_{2n-3,p} = -l_{2n-2,-p}$ for $q \in K_2(m')$ and $l_{2n-3,p} = -l_{2n-2,-p+1}$ for $q \in K_4(m')$, we get (5.1).

(2) Since $\tau_{0,-m'}\tau_{-m',0}Q = \tau_{-m',-m'}Q \in GT(\lambda)$, the equation (4.22), with Q replaced by $\tau_{-m',0}Q \in GT(\lambda)$ and with $j = -m'$, holds. Since $\tau_{0,-j'}\tau_{-m',0}Q \notin GT(\lambda)$ for $j' \in K_1(m') \cup K_5(m')$ and $\tau_{i',0}\tau_{-m',0}Q \notin GT(\lambda)$ for $i' \in [-m'+1,-m+1] \cup [m'+1,n-1]$, the terms in the right hand of (4.22) vanish except for the second line, the $(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,0})c(\tau_{-m',0}Q;a)$ term in the first line and $c(\tau_{m',0}\tau_{-m',0}Q;a)$ term in the third line. Thus we get (5.2).

(3) For this Q, $\tau_{i,-m'}Q \in GT(\lambda)$ if and only if $i = -m'$. (5.2) is the equation (4.15) applied to $j = -m'$.

(4) Consider the equation (4.22) applied to $j = k$. Since $\tau_{0,-j'}Q \notin GT(\lambda)$ if $j' \in K_1(m') \cup K_5(m')$ and $\tau_{i',0}Q \notin GT(\lambda)$ if $i' \in [-m'+1,-m+1] \cup [m', n-1]$, the terms in the right hand of (4.22) vanish except for the second line and the $(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,0})c(Q;a)$ term in the first line. Thus we get (5.4). $\square$

Proposition 5.2. *If $Q \in GT(\lambda)$ satisfies* (4.20), (4.23) *and* (4.24), *then $c(Q;a)$ is a solution of*

$$\left\{(\mathcal{D}_1^{\mp}(Q) - d_{m'}(Q))^2 - (\mathcal{D}_2^{\pm}(Q) \pm l_{2n-1,n})^2 - \left(\frac{\eta_1}{a_1}\right)^2\right\} c(Q;a) = 0. \tag{5.5}$$

Proof. Suppose $q_{2n-2,m'} = q_{2n-3,m'} > \lambda_{m'+1}$. Then (5.5) is obtained from (5.1), (5.2) and (5.3), by eliminating $c(\tau_{-m',0}Q;a)$ and $c(\tau_{-m',-m'}Q;a)$.

Next, suppose $q_{2n-2,m'} = q_{2n-3,m'} = \lambda_{m'+1}$. The equations (5.2) and (5.3), with Q replaced by $\tau_{m',m'}Q$, are difference equations for Q, $\tau_{m',m'}Q$ and $\tau_{0,m'}Q$. The equation (5.4), applied to $k = m'$, is a difference equation for Q and $\tau_{0,m'}Q$. By eliminating $\tau_{m',m'}Q$ and $\tau_{0,m'}Q$ from these, we get the equation (5.5). □

In order to deduce another differential equation, we prepare two identities.

Lemma 5.3. *For $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ and z,*

$$\sum_{i=1}^{k} \frac{\prod_{j=1}^{k-1}(x_i + y_j)\prod_{i'=1,\neq i}^{k}(z - x_{i'})}{\prod_{i'=1,\neq i}^{k}(x_i - x_{i'})} = \prod_{i=1}^{k-1}(z + y_i). \tag{5.6}$$

$$\sum_{i=1}^{k} \frac{\prod_{j=1}^{k}(x_i + y_j)}{(z + x_i)\prod_{i'=1,\neq i}^{k}(x_i - x_{i'})} = 1 - \frac{\prod_{i=1}^{k}(z - y_i)}{\prod_{i=1}^{k}(z + x_i)}. \tag{5.7}$$

Proof. (5.6) The left hand is a polynomial in z of degree $k-1$, which we denote by $f(z)$. For $l \in [1,k]$, $f(x_l) = \prod_{i=1}^{k-1}(x_l + y_i)$. On the other hand, the polynomial $\prod_{i=1}^{k-1}(z+y_i)$ of degree $k-1$ has the same values at k points x_l, so they are identical.

(5.7) Let $g(z) :=$ (left hand) $\times \prod_{i=1}^{k}(z+x_i)$, which is a polynomial in z of degree $k-1$. For $l \in [1,k]$, $g(-x_l) = (-1)^{k-1}\prod_{i=1}^{k}(x_l + y_i)$. On the other hand, the polynomial $\prod_{i=1}^{k}(z + x_i) - \prod_{i=1}^{k}(z - y_i)$ of degree $k-1$ has the same values at k points $z = -x_l$, so they are identical. □

Suppose $Q \in GT(\lambda)$ satisfies (4.20), (4.23), (4.24) and $q_{2n-2,m-1} < \lambda_{m-1}$. In this case, $K_6(m')$ is not empty since $m-1 \in K_6(m')$. The equation (4.14), applied to this Q and $j = m-1$, is

(5.8)
$$\left\{\mathcal{D}_2^{\pm}(Q) \mp \Lambda_{n+1} + \frac{\eta_2 a_1}{2a_2}(\iota A_0(Q) \mp 1)\right\} c(Q;a) + \iota\frac{\eta_2 a_1}{2a_2}\sum_{k \in K_6(m')} A_k(Q)c(\tau_{0,k}Q;a)$$
$$= 0.$$

Let $j \in K_6(m')$. After applying $\prod_{p \in K_6(m')\setminus\{j\}}(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,p} + 1)$ to both sides of (5.8), use (4.29) and (5.4). Then we get

(5.9)
$$-\iota\frac{\eta_2 a_1}{2a_2}A_j(Q)\left(\prod_{p \in K_6(m')\setminus\{j\}}(l_{2n-2,j} - l_{2n-2,p})\right) c(\tau_{0,j}Q;a)$$
$$= \left[\prod_{p \in K_6(m')\setminus\{j\}}(\mathcal{D}_2^{\pm}(Q) - l_{2n-2,p} + 1)\left\{\mathcal{D}_2^{\pm}(Q) \mp \Lambda_{n+1} + \frac{\eta_2 a_1}{2a_2}(\iota A_0(Q) \mp 1)\right\}\right.$$

$$
\begin{aligned}
&\pm\,\iota\frac{\eta_2 a_1}{2a_2}\sum_{k\in K_6(m')} A_k(Q)\iota B_{-k}(-k,\tau_{0,k}Q)\frac{\prod_{p\in J(m')\cup\{0\}}(l_{2n-2,k}-l_{2n-2,-p})}{\prod_{p\in J(m')\cup\{-m'\}}(l_{2n-2,k}+l_{2n-3,p})}\\
&\qquad\times\frac{\prod_{p\in K_6(m')\setminus\{j\}}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,p})-\prod_{p\in K_6(m')\setminus\{j\}}(l_{2n-2,k}-l_{2n-2,p})}{\mathcal{D}_2^{\pm}(Q)-l_{2n-2,k}}\\
&\qquad\times\left(\mp\frac{l_{2n-1,n}}{l_{2n-2,k}}\mathcal{D}_2^{\pm}(Q)+\mathcal{D}_1^{\mp}(Q)-d_{m'}(Q)-l_{2n-2,k}\right)\Bigg]c(Q;a).
\end{aligned}
$$

Let

$$
\begin{aligned}
K_7(m') := \{p\in[m-1,m'-1] | p+1\in K_6(m')\}&\\
\cup\,\{p\in[-n+1,-m'-1] | p\in K_6(m')\}.&
\end{aligned}
\tag{5.10}
$$

Note that $\#K_6(m')=\#K_7(m')+1$ if $q_{2n-2,m-1}<\lambda_{m-1}$. This is because $m-1\in K_6(m')$ if $q_{2n-2,m-1}<\lambda_{m-1}$, while there is no element in $K_7(m')$ corresponding to $m-1\in K_6(m')$. It is not hard to see that

$$
\iota A_0(Q)=\frac{\prod_{p=1}^{n-1}l_{2n-3,p}\prod_{p=1}^{n}l_{2n-1,p}}{\prod_{p=1}^{n-1}l_{2n-2,p}(l_{2n-2,p}-1)}\frac{\prod_{p=1}^{m-2}(-l_{2n-2,p})}{\prod_{p=1}^{m-1}(-l_{2n-1,p})}=\frac{l_{2n-1,n}\prod_{p\in K_7(m')}l_{2n-3,p}}{\prod_{p\in K_6(m')}l_{2n-2,p}}
$$

and

$$
\begin{aligned}
\iota A_k(Q)\,\iota B_{-k}(-k,\tau_{0,k}Q)&\frac{\prod_{p\in I_1(m')}(l_{2n-2,k}-l_{2n-2,-p})}{\prod_{p\in I_2(m')}(l_{2n-2,k}+l_{2n-3,p})}\\
&=\frac{\prod_{p\in K_7(m')}(l_{2n-2,k}-l_{2n-3,p})}{\prod_{p\in K_6(m')\setminus\{k\}}(l_{2n-2,k}-l_{2n-2,p})}.
\end{aligned}
$$

It follows that the last term in (5.9) is $\pm\eta_2 a_1/2a_2$ times

$$
\begin{aligned}
&\sum_{k\in K_6(m')}\frac{\prod_{p\in K_7}(l_{2n-2,k}-l_{2n-3,p})}{\prod_{p\in K_6(m')\setminus\{k\}}(l_{2n-2,k}-l_{2n-2,p})}\\
&\quad\times\frac{\prod_{p\in K_6(m')\setminus\{j\}}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,p})-\prod_{p\in K_6(m')\setminus\{j\}}(l_{2n-2,k}-l_{2n-2,p})}{\mathcal{D}_2^{\pm}(Q)-l_{2n-2,k}}\\
&\quad\times\left(\left(1\mp\frac{l_{2n-1,n}}{l_{2n-2,k}}\right)(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,k})+\mathcal{D}_1^{\mp}(Q)-d_{m'}(Q)-\mathcal{D}_2^{\pm}(Q)\mp l_{2n-1,n}\right)\\
&=\left(\sum_{k\in K_6(m')}\frac{l_{2n-2,k}\mp l_{2n-1,n}}{l_{2n-2,k}}\frac{\prod_{p\in K_7(m')}(l_{2n-2,k}-l_{2n-3,p})}{\prod_{p\in K_6(m')\setminus\{k\}}(l_{2n-2,k}-l_{2n-2,p})}\right)\\
&\quad\times\prod_{p\in K_6(m')\setminus\{j\}}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,p})\\
&\quad+\sum_{k\in K_6(m')}\frac{\prod_{p\in K_7(m')}(l_{2n-2,k}-l_{2n-3,p})}{\prod_{p\in K_6(m')\setminus\{k\}}(l_{2n-2,k}-l_{2n-2,p})}\frac{\prod_{p\in K_6(m')\setminus\{k\}}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,p})}{\mathcal{D}_2^{\pm}(Q)-l_{2n-2,j}}\\
&\quad\times\left(\mathcal{D}_1^{\mp}(Q)-d_{m'}(Q)-\mathcal{D}_2^{\pm}(Q)\mp l_{2n-1,n}\right)\\
&\quad-\prod_{p\in K_7(m')}(l_{2n-2,j}-l_{2n-3,p})\frac{1}{\mathcal{D}_2^{\pm}(Q)-l_{2n-2,j}}
\end{aligned}
$$

$$\times\left(\left(1\mp\frac{l_{2n-1,n}}{l_{2n-2,j}}\right)(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,j})+\mathcal{D}_1^{\mp}(Q)-d_{m'}(Q)-\mathcal{D}_2^{\pm}(Q)\mp l_{2n-1,n}\right)$$

$$=\left(1\mp\frac{l_{2n-1,n}\prod_{p\in K_7(m')}l_{2n-3,p}}{\prod_{p\in K_6(m')}l_{2n-2,p}}\right)\left(\prod_{p\in K_6(m')\setminus\{j\}}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,p})\right)$$

$$+\frac{\prod_{p\in K_7(m')}(\mathcal{D}_2^{\pm}(Q)-l_{2n-3,p})-\prod_{p\in K_7(m')}(l_{2n-2,j}-l_{2n-3,p})}{\mathcal{D}_2^{\pm}(Q)-l_{2n-2,j}}$$

$$\times\left(\mathcal{D}_1^{\mp}(Q)-d_{m'}(Q)-\mathcal{D}_2^{\pm}(Q)\mp l_{2n-1,n}\right)$$

$$-\frac{l_{2n-2,j}\mp l_{2n-1,n}}{l_{2n-2,j}}\prod_{p\in K_7(m')}(l_{2n-2,j}-l_{2n-3,p}).$$

Here, we use Lemma 5.3 to get the last equality. Then (5.9) becomes the next equation.

Lemma 5.4. *If $Q\in GT(\lambda)$ satisfies* (4.20), (4.23) *and* (4.24), *then for $j\in K_6(m')$,*

(5.11)

$$-\iota\frac{\eta_2a_1}{2a_2}A_j(Q)\left(\prod_{p\in K_6(m')\setminus\{j\}}(l_{2n-2,j}-l_{2n-2,p})\right)c(\tau_{0,j}Q;a)$$

$$=\left(\prod_{p\in K_6(m')\setminus\{j\}}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,p}+1)\right)(\mathcal{D}_2^{\pm}(Q)\mp\Lambda_{n+1})c(Q;a)$$

$$\pm\frac{\eta_2a_1}{2a_2}\Bigg[\frac{\prod_{p\in K_7(m')}(\mathcal{D}_2^{\pm}(Q)-l_{2n-3,p})-\prod_{p\in K_7(m')}(l_{2n-2,j}-l_{2n-3,p})}{\mathcal{D}_2^{\pm}(Q)-l_{2n-2,j}}$$

$$\times\left(\mathcal{D}_1^{\mp}(Q)-d_{m'}(Q)-\mathcal{D}_2^{\pm}(Q)\mp l_{2n-1,n}\right)$$

$$-\frac{l_{2n-2,j}\mp l_{2n-1,n}}{l_{2n-2,j}}\prod_{p\in K_7(m')}(l_{2n-2,j}-l_{2n-3,p})\Bigg]c(Q;a)$$

Proposition 5.5. *If $Q\in GT(\lambda)$ satisfies* (4.20), (4.23), (4.24) *and $q_{2n-2,m-1}<\lambda_{m-1}$, then $c(Q;a)$ is a solution of*

(5.12)

$$\Bigg\{(\mathcal{D}_2^{\pm}(Q)\mp\Lambda_{n+1})\prod_{k\in K_6(m')}(\mathcal{D}_2^{\pm}(Q)-l_{2n-2,k}+1)$$

$$\pm\frac{\eta_2a_1}{2a_2}(\mathcal{D}_1^{\mp}(Q)-d_{m'}(Q)-\mathcal{D}_2^{\pm}(Q)\mp l_{2n-1,n})\prod_{p\in K_7(m')}(\mathcal{D}_2^{\pm}(Q)-l_{2n-3,p})\Bigg\}$$

$$\times c(Q;a)$$

$$=0.$$

If $m'=m-1$ and $q_{2n-2,m-1}=\lambda_{m-1}$, then $c(Q;a)$ satisfies the equation (5.12) *with $K_6(m')$ replaced by $K_6(m-1)\cup\{m-1\}$.*

Proof. Suppose $q_{2n-2,m-1}<\lambda_{m-1}$. In this case $K_6(m')$ is not empty since $m-1\in K_6(m')$. Differentiate (5.11) by $\mathcal{D}_2^{\pm}(Q)-l_{2n-2,j}+1$, and use (5.4). Then we

get the equation (5.12). Suppose $m' = m-1$ and $q_{2n-2,m-1} = \lambda_{m-1}$. Then $K_6(m-1)(\tau_{-m+1,-m+1}Q) = K_6(m-1)(Q) \cup \{m-1\}$. It follows that the equation (5.11) holds for $j = m-1$, if we replace Q by $\tau_{-m+1,-m+1}Q$. From this equation and the equations (5.1) and (5.3), applied to $m' = m-1$, we obtain (5.12) by eliminating $c(\tau_{-m+1,0}Q; a)$ and $c(\tau_{-m+1,-m+1}Q; a)$. $\square$

5.2. **Solutions of** (5.5) **and** (5.12)**.** Suppose $\Lambda \in \Xi_{m,\pm}$, $m = 2, \dots, n$, and $m' \in [m-1, n-1]$. Assume $Q \in GT(\lambda)$ satisfies (4.20), (4.23) and (4.24). Define j_i and k_i by

$$\{j_3, \dots, j_{N_1}\} = K_6(m') \cap [m, m'], \quad 0 < j_3 < \dots < j_{N_1},$$
$$\{k_{N_1+1}, \dots, k_{N_2}\} = K_6(m') \cap [-n+1, -m'-1], \quad k_{N_1+1} < \dots < k_{N_2} < 0.$$

If $K_6(m') \cap [m, m']$ (resp. $K_6(m') \cap [-n+1, -m'-1]$) is empty, then we set $N_1 = 2$ (resp. $N_2 = N_1$).

When $m' \geq m$, define

$$\alpha_p = \alpha_p(m', Q) := \begin{cases} \pm\Lambda_n \pm \Lambda_{n+1}, & \text{for } p = 1, \\ \pm\Lambda_n + l_{2n-2,m-1} - 1, & \text{for } p = 2, \\ \pm\Lambda_n + l_{2n-2,j_p} - 1, & \text{for } 3 \leq p \leq N_1, \\ \pm\Lambda_n + l_{2n-2,k_p} - 1, & \text{for } N_1 + 1 \leq p \leq N_2. \end{cases}$$

Suppose $m' = m-1$. In this case, (i) $K_6(m') \cap [m, m']$ is empty, and (ii) the inequality relation of $\pm\Lambda_{n+1}$ and $l_{2n-2,m-1} - 1$ depends on the value of $q_{2n-2,m-1}$. For these reasons, we define

$$\alpha_p = \alpha_p(m', Q) := \begin{cases} \pm\Lambda_n + \max\{\pm\Lambda_{n+1}, l_{2n-2,m-1} - 1\}, & \text{for } p = 1, \\ \pm\Lambda_n + \min\{\pm\Lambda_{n+1}, l_{2n-2,m-1} - 1\}, & \text{for } p = 2, \\ \pm\Lambda_n + l_{2n-2,k_p} - 1, & \text{for } 3 \leq p \leq N_2. \end{cases}$$

Remark 5.6. When $m' \in K_6(m')$, $l_{2n-2,m'}$ corresponds to α_{N_1} except for the case $m' = m-1$ and $l_{2n-2,m-1} - 1 \geq \pm\Lambda_{n+1}$.

Consider the number $l_{2n-3,p}$ for $p \in K_7(m')$. By definition (5.10) of $K_7(m')$,

$$K_7(m') = \{j_3 - 1, \dots, j_{N_1} - 1\} \cup \{k_{N_1+1}, \dots, k_{N_2}\}.$$

Define

$$\beta_p = \beta_p(m', Q) := \begin{cases} \pm\Lambda_n + l_{2n-3,j_p-1}, & \text{for } 3 \leq p \leq N_1, \\ \pm\Lambda_n + l_{2n-3,k_p}, & \text{for } N_1 + 1 \leq p \leq N_2. \end{cases}$$

Lemma 5.7. *These numbers satisfy*

$$\begin{aligned} \alpha_1 \geq& \alpha_2 \geq \beta_3 > \alpha_3 \geq \dots > \alpha_{N_1-1} \geq \beta_{N_1} > \alpha_{N_1} \\ &> 0 \geq \beta_{N_1+1} > \alpha_{N_1+1} \geq \beta_{N_1+2} > \dots > \alpha_{N_2-1} \geq \beta_{N_2} > \alpha_{N_2}, \end{aligned}$$

and the difference between β_p and α_p is at least two.

Proof. Recall Remark 3.10 and the condition (4.24). If $p \in K_6(m') \cap [m-1, m'-1]$, then $l_{2n-2,p} - 1 = \lambda_{p+1} + n - p - 1 = \Lambda_{p+1}$ since $q_{2n-2,p} = \lambda_{p+1}$. If $p \in K_6(m') \cap [-n-1, -m'-1]$, then $l_{2n-2,p} - 1 = -l_{2n-2,-p} = -(\lambda_{-p} + n + p) = -\Lambda_{-p}$ since $q_{2n-2,-p} = \lambda_{-p}$. We know $\Lambda_{m'} > l_{2n-2,m'} - 1 \geq \Lambda_{m'+1} - 1$ since $\lambda_{m'} \geq q_{2n-2,m'} \geq \lambda_{m'+1}$. It follows that

$$\alpha_1 \geq \alpha_2 > \dots > \alpha_{N_1} > 0 > \alpha_{N_1+1} > \dots > \alpha_{N_2},$$

since the Harish-Chandra parameter $\Lambda = \sum_{i=1}^{n+1} \Lambda_i e_i \in \Xi_{m,\pm}$ satisfies

$$\Lambda_1 > \cdots > \Lambda_{m-1} > \pm\Lambda_{n+1} > \Lambda_m > \cdots > \Lambda_{n-1} > |\Lambda_n|.$$

Next, consider the numbers β_j. If $3 \leq p \leq N_1$, then $l_{2n-2,j_p-1} - 1 \geq l_{2n-3,j_p-1} > l_{2n-2,j_p} > l_{2n-2,j_p} - 1$ since $q_{2n-2,j_p-1} \geq q_{2n-3,j_p-1} > q_{2n-2,j_p}$. If $N_1 + 1 \leq p \leq N_2$, then $l_{2n-2,k_p-1} - 1 \geq l_{2n-3,k_p} > l_{2n-2,k_p} > l_{2n-2,k_p} - 1$ since $q_{2n-2,-k_p} > q_{2n-3,-k_p} \geq q_{2n-2,-k_p+1}$. It follows that $\alpha_{p-1} \geq \beta_p > \alpha_p$, and the difference between β_p and α_p is at least two. $\square$

In order to rewrite (5.5) and (5.12) in a convenient form, let

$$(5.13)\quad t_1 := \frac{\eta_1}{a_1}, \quad t_2 := \mp\frac{\eta_1\eta_2}{2a_2}, \quad \partial_{t_i} := t_i\frac{\partial}{\partial t_i} \ (i = 1, 2) \quad \text{and}$$

$$(5.14)\quad n(Q, m'; a) := a_1^{-n+1+d_{m'}(Q)} a_2^{-\sum_{p=1}^{m-1} l_{2n-1,p} + \sum_{p=1}^{m-2} l_{2n-2,p} \mp(\Lambda_n+\lambda_{n+1}) - m+2} \times \exp\left(\pm\frac{\eta_2 a_1}{2a_2}\right).$$

Note that $t_1 > 0$ since $\eta_1 > 0$ and $a_1 > 0$. We have $\partial_i = -\partial_{t_i}$ $(i = 1, 2)$ and $\eta_2 a_1/2a_2 = \mp t_2/t_1$. Since $l_{2n-2,n} = \lambda_n = \Lambda_n$, we have the following proposition.

Proposition 5.8. *Assume $Q \in GT(\lambda)$ satisfies* (4.20), (4.23) *and* (4.24). *Define*

$$f(Q, m'; t) := n(Q, m'; a)^{-1} c(Q; a).$$

Then, the differential equations (5.5) *and* (5.12) *are expressed as*

$$(5.15)\qquad (\partial_{t_1}^2 - \partial_{t_2}^2 - t_1^2) f(Q, m'; t) = 0,$$

$$(5.16)\qquad \left\{\prod_{p=1}^{N_2}(\partial_{t_2} + \alpha_p) + \frac{t_2}{t_1}(\partial_{t_1} - \partial_{t_2})\prod_{p=3}^{N_2}(\partial_{t_2} + \beta_p)\right\} f(Q, m'; t) = 0.$$

Proposition 5.9. *Let C_j be a loop starting and ending at $+\infty$, crossing the real axis at $-\alpha_j - 1 < s < -\alpha_j$, and encircling all poles of $\Gamma(-\alpha_p - s)$, $p = j, \ldots, N_2$, once in the negative direction, but none of the poles of $\Gamma(\beta_p + s)$, $p = 3, \ldots, j$, i.e. encircling the half real axis $\{x + 0i \in \mathbb{C} | x \geq -\alpha_j\}$ once in the negative direction. Define*

$$\begin{Bmatrix} f_1^K(Q, m'; t) \\ f_1^I(Q, m'; t) \end{Bmatrix} := \frac{1}{2\pi\iota}\int_{C_1} \frac{\prod_{p=1}^{N_2}\Gamma(-\alpha_p - s)}{\prod_{p=3}^{N_2}\Gamma(1 - \beta_p - s)} \begin{Bmatrix} t_2^s\, K_{-s}(t_1) \\ (-t_2)^s\, I_{-s}(t_1) \end{Bmatrix} ds,$$

and, for $j = 2, \ldots, N_2$, define

$$\begin{Bmatrix} f_j^K(Q, m'; t) \\ f_j^I(Q, m'; t) \end{Bmatrix} := \frac{1}{2\pi\iota}\int_{C_j} \frac{\prod_{p=j}^{N_2}\Gamma(-\alpha_p - s)\prod_{p=3}^{j}\Gamma(\beta_p + s)}{\prod_{p=1}^{j-1}\Gamma(1 + \alpha_p + s)\prod_{p=j+1}^{N_2}\Gamma(1 - \beta_p - s)} \begin{Bmatrix} (-t_2)^s\, K_{-s}(t_1) \\ t_2^s\, I_{-s}(t_1) \end{Bmatrix} ds.$$

Here, $K_\mu(z)$ and $I_\nu(z)$ are modified Bessel functions (cf. [4]*). Then these integrals absolutely converge for $(t_1, t_2) \in \mathbb{C}^2 \setminus (\{t_1 = 0\} \cup \{t_2 = 0\})$, and they form a basis of the solution space of the system of equations* (5.15) *and* (5.16). *Moreover, if*

$\alpha_1 \neq \alpha_2$, the leading terms of these functions at $t_2 = 0$ are

$$\begin{Bmatrix} f_1^K \\ f_1^I \end{Bmatrix} \quad \cdots \quad \frac{\prod_{p=2}^{N_2} \Gamma(\alpha_1 - \alpha_p)}{\prod_{p=3}^{N_2} \Gamma(\alpha_1 - \beta_p + 1)} \begin{Bmatrix} t_2^{-\alpha_1} K_{\alpha_1}(t_1) \\ (-t_2)^{-\alpha_1} I_{\alpha_1}(t_1) \end{Bmatrix}, \quad \textit{and}$$

$$\begin{Bmatrix} f_j^K \\ f_j^I \end{Bmatrix} \quad \cdots \quad \frac{\prod_{p=j+1}^{N_2} \Gamma(\alpha_j - \alpha_p) \prod_{p=3}^{j} \Gamma(\beta_p - \alpha_j)}{\prod_{p=1}^{j-1} \Gamma(1 + \alpha_p - \alpha_j) \prod_{p=j+1}^{N_2} \Gamma(\alpha_j - \beta_p + 1)} \begin{Bmatrix} (-t_2)^{-\alpha_j} K_{\alpha_j}(t_1) \\ t_2^{-\alpha_j} I_{\alpha_j}(t_1) \end{Bmatrix}$$

if $j \geq 2$, respectively. If $\alpha_1 = \alpha_2$, then the leading terms of f_1^K, f_1^I are complicated.

Proof. At a generic point, the solution space of the system of equations (5.15) and (5.16) is $2N_2$ dimensional. It is easy to verify that these integrals formally satisfy the equations (5.15) and (5.16).

Let $s = u + v\iota$, where u is a very large positive real number and v is a non-zero finite real number. We will see the asymptotic behavior of the integrands when $u \to \infty$. In the following of this proof, $C_j(v)$'s are positive constants which depend on v.

By the asymptotic expansion of the gamma function, we know that, if $|\arg s| < \pi$ and $|s|$ is large,

$$\Gamma(s + a) = s^{s-1/2+a} e^{-s} \sqrt{2\pi} \times O(1), \quad |s| \to \infty.$$

Hence we have

$$\left| \frac{\prod_{p=1}^{N_2} \Gamma(-\alpha_p - s)}{\prod_{p=3}^{N} \Gamma(1 - \beta_p - s)} \right|, \left| \frac{\prod_{p=j}^{N_2} \Gamma(-\alpha_p - s) \prod_{p=3}^{j} \Gamma(\beta_p + s)}{\prod_{p=1}^{j-1} \Gamma(1 + \alpha_p + s) \prod_{p=j+1}^{N_2} \Gamma(1 - \beta_p - s)} \right|$$
$$\leq C_1(v) \exp \left\{ \left(-2u - \sum_{p=1}^{N_2} \alpha_p + \sum_{p=3}^{N_2} \beta_p - N_2 + 1 \right) \log |u + v\iota| + 2u \right\}.$$

Since

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\, \Gamma(\nu + n + 1)} \quad \text{and} \quad K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi},$$

and t_1 is a positive real number, we have

$$|K_{-u-v\iota}(t_1)|, |I_{-u-v\iota}(t_1)| \leq C_2(v) \exp \left\{ \left(u + \frac{1}{2} \right) \log |-u - v\iota| - u(\log \frac{t_1}{2} + 1) \right\}.$$

Next,

$$|(\pm t_2)^{u+v\iota}| \leq C_3(v) |t_2|^u.$$

Thus the integrals in this proposition absolutely converge. The leading terms are obtained by the residue theorem. They imply that f_j^L, $j = 1, \ldots, N_2$, $L = K, I$, are linearly independent. □

5.3. **Solutions of the whole differential-difference equations.** We consider whether the solutions of scalar equations obtained in the previous subsection satisfy the whole differential-difference equations $\mathcal{D}_{\tilde{\lambda},\eta} \phi = 0$ or not.

The equation (5.2), with Q replaced by $\tau_{m',m'} Q$, expresses $c(\tau_{m',m'} Q; a)$ as a sum of differentials of $c(Q; a)$ and $c(\tau_{0,m'} Q; a)$. The equation (5.11), with $j = m'$, expresses $c(\tau_{0,m'} Q; a)$ as a differential of $c(Q; a)$. By (i) eliminating $c(\tau_{0,m'} Q; a)$ from these equations, (ii) changing the independent variables from a_i to t_i and the dependent variables from $c(Q; a)$ to $f(Q; a)$, and (iii) simplifying the equation so obtained by using (5.15) and (5.16), we get the following shift operator.

Proposition 5.10. *Suppose $Q \in GT(\lambda)$ satisfies* (4.20), (4.23) *and* (4.24), *and suppose $\tau_{m',m'}Q \in GT(\lambda)$. Then*

$$
\begin{aligned}
&f(\tau_{m',m'}Q, m'; t) = (\text{nonzero constant}) \times S_1(m', Q) f(Q, m'; t),\\
&S_1(m', Q)\\
&:= \frac{1}{t_1 t_2}(\partial_{t_1} + \partial_{t_2}) \prod_{p=1, \neq N_1}^{N_2} (\partial_{t_2} + \alpha_p) + \frac{\prod_{p=3}^{N_2}(\partial_{t_2} + \beta_p) - \prod_{p=3}^{N_2}(\beta_p - \alpha_{N_1} - 1)}{\partial_{t_2} + \alpha_{N_1} + 1}.
\end{aligned}
$$

For notational convenience, let $Q' := \tau_{m',m'}Q$ and denote $\alpha_j(Q')$, $\beta_j(Q')$ by α'_j, β'_j, respectively. As before, α_j, β_j mean $\alpha_j(Q)$, $\beta_j(Q)$, respectively. Let $N_1 = \#(K_6(m')(Q) \cap [m, m']) + 2$ and $N_2 = \#(K_6(m')(Q) \cap [-n+1, -m'-1]) + N_1$, i.e. they are the numbers N_1, N_2 used in the previous section, defined for Q (not Q').

Suppose $q_{2n-2,m'} = q_{2n-3,m'} < q_{2n-3,m'-1} = q_{2n-4,m'-1}$. Then $K_6(m')(Q') = K_6(m')(Q)$. Therefore,

$$
\begin{aligned}
&\alpha'_j = \alpha_j, \quad \text{for } j = 1, \ldots, N_1 - 1, N_1 + 1, \ldots, N_2,\\
&\alpha'_{N_1} = \alpha_{N_1} + 1,\\
&\beta'_j = \beta_j, \quad \text{for } j = 3, \ldots, N_2.
\end{aligned}
$$

In this case, it is easy to verify that

$$
S_1(m', Q) f_j^L(Q, m'; t) = \prod_{p=1}^{N_2} (\beta_p - \alpha_{N_1} - 1) \times \begin{cases} f_j^L(Q', m'; t) & \text{if } j \leq N_1,\\ -f_j^L(Q', m'; t) & \text{if } j > N_1, \end{cases}
$$

for $j = 1, 2, \ldots, N_2$ and $L = K$ or I.

Suppose $m' \geq m$ and $q_{2n-2,m'} = q_{2n-3,m'} = q_{2n-3,m'-1} = q_{2n-4,m'-1}$. Then $K_6(m')(Q') = K_6(m')(Q) \setminus \{m'\}$. It follows that

$$
\begin{aligned}
&\alpha'_j = \alpha_j, \quad \text{for } j = 1, \ldots, N_1 - 1, \qquad \alpha'_j = \alpha_{j+1}, \quad \text{for } j = N_1, \ldots, N_2 - 1,\\
&\beta'_j = \beta_j, \quad \text{for } j = 3, \ldots, N_1 - 1. \qquad \beta'_j = \beta_{j+1}, \quad \text{for } j = N_1, \ldots, N_2 - 1.
\end{aligned}
$$

In this case, for $L = K, I$,

$$
S_1(m', Q) f_j^L(Q, m'; t) = \prod_{p=1}^{N} (\beta_p - \alpha_{N_1} - 1) \times \begin{cases} f_j^L(Q', m'; t), & \text{for } j < N_1\\ -f_{j-1}^L(Q', m'; t), & \text{for } j > N_1. \end{cases}
$$

On the other hand, the non-zero functions $S_1(m', Q) f_{N_1}^L(Q, m'; t)$, $L = K, I$, are not solutions of (5.15) and (5.16). Since $(\tau_{m',m'})^{q_{2n-4,m'-1} - \lambda_{m'+1}} Q^-_{m'} = Q^+_{m'}$, we get the following proposition.

Proposition 5.11. *For $m' \geq m$ and $L = K, I$,*

$$
\begin{aligned}
&\prod_{p=0}^{q_{2n-4,m'-1} - \lambda_{m'+1} - 1} S_1(m', \tau^p_{m',m'} Q) f_j^L(Q^-_{m'}, m'; t)\\
&\qquad = \begin{cases} (\text{non-zero constant}) \times f_j^L(Q^+_{m'}, m'; t), & \textit{if } j < N_1,\\ \textit{not a solution of (5.15) and (5.16) for } Q^+_{m'} & \textit{if } j = N_1,\\ (\text{non-zero constant}) \times f_{j-1}^L(Q^+_{m'}, m'; t), & \textit{if } j > N_1. \end{cases}
\end{aligned}
$$

Suppose $m' \geq m$ and $\lambda_{m'} > q_{2n-4,m'-1}$. Consider $Q \in GT(\lambda)$ satisfying (4.20), (4.23) and

$$\begin{cases} q_{2n-2,p} = \lambda_{p+1} \text{ for } p \in [m-1, m'-1], \\ q_{2n-3,p} = q_{2n-4,p} \text{ for } p \in [m-1, m'-2], \\ q_{2n-2,p} = \lambda_p \text{ for } p \in [m'+1, n-1], \\ q_{2n-3,p} = q_{2n-4,p-1} \text{ for } p \in [m', n-1], \\ q_{2n-2,m'} = q_{2n-3,m'-1} \in [q_{2n-4,m'-1}, \lambda_{m'}]. \end{cases} \tag{5.17}$$

Note that, if $q_{2n-2,m'} = q_{2n-3,m'-1} = q_{2n-4,m'-1}$, then such Q is $Q^+_{m'}$ defined in Definition 4.22. If $q_{2n-2,m'} = q_{2n-3,m'-1} = \lambda_{m'}$, then such Q is $Q^-_{m'-1}$.

For $Q \in GT(\lambda)$ satisfying (4.20), (4.23) and (5.17), the equation (4.15), with $j = m'$, is

$$c(\tau_{m'-1,m'}Q; a) = (\text{nonzero constant}) \times \frac{a_1}{\eta_1}(\mathcal{D}^{\pm}_2(Q) + l_{2n-2,m'})c(Q; a).$$

As before, let

$$\begin{aligned} f(Q^+_{m'}, m'; t) &= n(Q^+_{m'}, m', a)^{-1}c(Q^+_{m'}; a), \\ f(Q^-_{m'-1}, m'-1; t) &= n(Q^-_{m'-1}, m'-1, a)^{-1}c(Q^-_{m'-1}; a). \end{aligned}$$

Since $(\tau_{m'-1,m'})^{\lambda_{m'}-q_{2n-4,m'-1}}Q^+_{m'} = Q^-_{m'-1}$, we obtain a shift operator

$$\begin{aligned} f(Q^-_{m'-1}, m'-1; t) &= (\text{non-zero constant}) \times S_2(m')f(Q^+_{m'}, m'; t), \\ S_2(m') &:= \prod_{q=q_{2n-4,m'-1}}^{\lambda_{m'}-1} (\partial_{t_2} \pm \Lambda_n - q - n + m'). \end{aligned}$$

Recall that, for

$$K_6(m')(Q^+_{m'}) = \{m-1\} \cup \{j_3, \ldots, j_{N_1}\} \cup \{k_{N_1+1}, \ldots, k_{N_2}\},$$

$K_7(m')(Q^+_{m'})$ is defined to be

$$K_7(m')(Q^+_{m'}) = \{j_3 - 1, \ldots, j_{N_1} - 1\} \cup \{k_{N_1+1}, \ldots, k_{N_2}\}.$$

Suppose $Q^-_{m'-1} \neq Q^+_{m'}$. Since $K_6(m'-1)(Q^-_{m'-1}) = K_6(m')(Q^+_{m'}) \cup \{-m'\}$, we have

$$\begin{aligned} K_6(m'-1)(Q^-_{m'-1}) &= \{m-1\} \cup \{j_3, \ldots, j_{N_1}\} \cup \{-m', k_{N_1+1}, \ldots, k_{N_2}\}, \\ K_7(m'-1)(Q^-_{m'-1}) &= \{j_3 - 1, \ldots, j_{N_1} - 1\} \cup \{-m', k_{N_1+1}, \ldots, k_{N_2}\}. \end{aligned}$$

For simplicity, denote $\alpha_j(Q^+_{m'})$, $\beta_j(Q^+_{m'})$ by α'_j, β'_j, and $\alpha_j(Q^-_{m'-1})$, $\beta_j(Q^-_{m'-1})$ by α''_j, β''_j, respectively. They are related as follows:

$$\begin{aligned} &\alpha''_p = \alpha'_p \quad \text{for } 1 \leq p \leq N_1, && \beta''_p = \beta'_p \quad \text{for } 3 \leq p \leq N_1, \\ &\alpha''_{N_1+1} = \pm\Lambda_n - \lambda_{m'} - n + m' && \beta''_{N_1+1} = \pm\Lambda_n - q_{2n-4,m'-1} - n + m' + 1 \\ &\alpha''_p = \alpha'_{p-1} \quad \text{for } N_1 + 2 \leq p \leq N_2 + 1, && \beta''_p = \beta'_{p-1} \quad \text{for } N_1 + 2 \leq p \leq N_2 + 1, \end{aligned}$$

and they are ordered as follows:

$$\begin{aligned} &\alpha'_1 > \alpha'_2 \geq \beta'_3 > \alpha'_3 \geq \cdots > \alpha'_{N_1-1} \geq \beta'_{N_1} > \alpha'_{N_1} \\ &\quad > 0 > \beta''_{N_1+1} > \alpha''_{N_1+1} \geq \beta'_{N_1+1} > \alpha'_{N_1+1} \geq \beta'_{N_1+2} > \cdots > \alpha'_{N_2-1} \geq \beta'_{N_2} > \alpha'_{N_2}. \end{aligned}$$

Since

$$\begin{aligned}&S_2(m')(\pm t_2)^s\\&\quad=\frac{\Gamma(\beta''_{N_1+1}+s)}{\Gamma(1+\alpha''_{N_1+1}+s)}(\pm t_2)^s=(-)^{\beta''_{N_1+1}-\alpha''_{N_1+1}-1}\frac{\Gamma(-\alpha''_{N_1+1}-s)}{\Gamma(1-\beta''_{N_1+1}-s)}(\pm t_2)^s,\end{aligned}$$

we have the following proposition.

Proposition 5.12. *For $L=K$ or I, $S_2(m')f_j^L(Q^+_{m'},m';t)$ is a non-zero constant multiple of*

(1) $f_j^L(Q^-_{m'-1},m'-1;t)$ *if* $j=1,\dots,N_1$,
(2) $f_{j+1}^L(Q^-_{m'-1},m'-1;t)$ *if* $j=N_1+1,\dots,N_2$.

Propositions 5.11, 5.12 enable us to judge whether f_j^L, $j=1,\dots,N_2$, $L=K,I$, generates the whole solution of $\mathcal{D}_{\tilde{\lambda},\eta}\phi=0$.

By Theorem 4.23, $c(Q^+_{n-1};a)$, therefore $f(Q^+_{n-1};t)$, determines all the $c(Q;a)$ containing the same $\mathbf{q}_1,\dots,\mathbf{q}_{2n-4}$ parts as Q^+_{n-1}. Let $\alpha_j=\alpha_j(Q^+_{n-1})$. This is the leading exponent of $f_j^L(Q^+_{n-1};t)$ at $t_2=0$. Since $\alpha_1=\pm(\Lambda_n+\Lambda_{n+1})$ and $\alpha_2=\pm\Lambda_n+l_{2n-2,m-1}-1$, alternative use of Proposition 5.11 and Proposition 5.12 implies that, if $j=1,2$ and $L=K,I$,

$$\begin{aligned}f_j^L(Q^+_{n-1},n-1;t)&\overset{\text{Prop.5.11}}{\longrightarrow}f_j^L(Q^-_{n-2},n-2;t)\overset{\text{Prop.5.12}}{\longrightarrow}f_j^L(Q^+_{n-2},n-2;t)\longrightarrow\dots\\&\dots\overset{\text{Prop.5.11}}{\longrightarrow}f_j^L(Q^-_{m-1},m-1;t)\end{aligned}$$

On the other hand, if $j\geq 3$, there exists $m'\in K_6(n-1)(Q^+_{n-1})\cap[m,n-2]$ such that $\alpha_j(Q^+_{n-1})=\pm\Lambda_n+l_{2n-2,m'}-1$. In this case,

$$\begin{aligned}&f_j^L(Q^+_{n-1},n-1;t)\overset{\text{Prop.5.11}}{\longrightarrow}f_j^L(Q^-_{n-2},n-2;t)\overset{\text{Prop.5.12}}{\longrightarrow}f_j^L(Q^+_{n-2},n-2;t)\longrightarrow\dots\\&\dots\overset{\text{Prop.5.12}}{\longrightarrow}f_j^L(Q^-_{m'},m';t)\overset{\text{Prop.5.11}}{\longrightarrow}\text{not a solution of (5.15) and (5.16) for }Q^+_{m'}.\end{aligned}$$

This implies that only $f_j^L(Q^+_{n-1},n-1;t)$, $j=1,2$, $L=K,I$, can generate a solution of the whole differential-difference equations. It follows that the constant C in Theorem 3.8 is at most four. In the special case when $m=n$, we can check the compatibility of the equations in Lemma 4.13. Therefore, this constant C, which is independent of m, is just four.

Theorem 5.13. *Let $\Lambda\in\Xi_{m,\pm}$, $m=2,\dots,n$. The functions $f_1^K(Q^+_{n-1},n-1;t)$, $f_1^I(Q^+_{n-1},n-1;t)$, $f_2^K(Q^+_{n-1},n-1;t)$, $f_2^I(Q^+_{n-1},n-1;t)$, with Q^+_{n-1} defined in Definition 4.22 completely determines the non-zero solutions of $\mathcal{D}_{\tilde{\lambda},\eta}\phi=0$.*

5.4. **Continuous Whittaker models.** So far, we have investigated the space $\mathrm{Hom}_{\mathfrak{g},K_{\mathbb{R}}}(\pi^*_\Lambda,C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$. Here, we specify the subspace of continuous intertwining operators $\mathrm{Hom}^\infty_{G_{\mathbb{R}}}((\pi^*_\Lambda)_\infty,C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$. Note that the latter space is isomorphic to $\mathrm{Wh}^\infty_{-\eta}(\pi^*_\Lambda)$.

Proposition 5.14. *Suppose $G_{\mathbb{R}}=Spin(2n,2)$. Let π_Λ be the discrete series representation with the Harish-Chandra parameter Λ.*

(1) *Suppose $\Lambda\in\Xi_{m,+}$ and $\Lambda'\in\Xi_{m,-}$, $m=2,\dots,n$. If $\mathrm{Wh}^\infty_{-\eta}(\pi^*_\Lambda)\neq\{0\}$, then $\mathrm{Wh}^\infty_{-\eta}(\pi^*_{\Lambda'})=\{0\}$.*
(2) *Let η and η' be non-degenerate unitary characters defined as in (4.6). Suppose $\eta_2\eta'_2<0$. If $\mathrm{Wh}^\infty_{-\eta}(\pi^*_\Lambda)\neq\{0\}$, then $\mathrm{Wh}^\infty_{-\eta'}(\pi^*_\Lambda)=\{0\}$.*

(3) *Suppose* $\Lambda \in \Xi_{m,\pm}$, $m = 2, \ldots, n$. *If* $\mathrm{Wh}^{\infty}_{-\eta}(\pi^*_\Lambda) \neq \{0\}$, *then*

$$\dim \mathrm{Wh}^{\infty}_{-\eta}(\pi^*_\Lambda) = \sum_{\substack{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-2} \geq \mu_{m-2} \geq \lambda_{m-1} \\ \lambda_m \geq \mu'_1 \geq \lambda_{m+1} \geq \cdots \geq \lambda_{n-1} \geq \mu'_{n-m} \geq |\lambda_n|}} \dim V^{Spin(2n-3,\mathbb{C})}_{(\mu_1,\ldots,\mu_{m-2},\mu'_1,\ldots,\mu'_{n-m})}.$$

Proof. For our case $G_\mathbb{R} = Spin(2n, 2)$, there are two principal nilpotent $G_\mathbb{R}$-orbits on $\mathfrak{g}_0$, and $W_{G_\mathbb{R}} \simeq \mathfrak{S}_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, (3) follows from Theorems 3.2, 3.8, Remark 3.10 and Theorem 5.13.

Via the Kostant-Sekiguchi correspondence ([12]), these two orbits correspond to the two nilpotent K-orbits on $\mathfrak{p}$ generated by $X_{-e_1 \pm e_{n+1}} + X_{e_n \mp e_{n+1}} + X_{-e_n \mp e_{n+1}}$ (see the proof of Proposition 3.5). Schmid and Vilonen ([11]) proved that the associated cycle of a Harish-Chandra $(\mathfrak{g}, K_\mathbb{R})$-module and the wave front cycle of it are related under the Kostant-Sekiguchi correspondence. It follows that, for $m = 2, \ldots, n$, the wave front set of the discrete series π_Λ with $\Lambda \in \Xi_{m,+}$ and that of $\pi_{\Lambda'}$ with $\Lambda' \in \Xi_{m,-}$ are different principal nilpotent $G_\mathbb{R}$-orbits. Therefore, (1) is a consequence of Theorem 3.2 (1). Recall the identification of a unitary character with an element of $\iota(\mathfrak{n}_0/[\mathfrak{n}_0, \mathfrak{n}_0])^* \subset \iota\mathfrak{g}_0^* \simeq \iota\mathfrak{g}_0$ (cf. §3.1). It is easy to check that η and η' are contained in different principal nilpotent $G_\mathbb{R}$-orbits multiplied by ι. Therefore, (2) also follows from Theorem 3.2 (1). □

By a theorem of Wallach ([15]), if $\psi \in \mathrm{Hom}^{\infty}_{G_\mathbb{R}}((\pi^*_\Lambda)_\infty, C^\infty(G_\mathbb{R}/N_\mathbb{R}; \eta))$, then $\psi(v)(g)$, $v \in (\pi^*_\Lambda)_\infty$, $g \in G_\mathbb{R}$, must be a moderate-growth function. We show that, if $\mp\eta_2 > 0$, then the function f^K_1 defined in Proposition 5.9 generates a rapidly decreasing Whittaker function.

Recall the definition (5.13) of t_1, t_2. Since $a_1, a_2 > 0$, $\eta_1 > 0$ and $\mp\eta_2 > 0$, we have $t_1 > 0$ and $t_2 > 0$. Let

$$S_3 = \prod_{p=3}^{N_2} \prod_{q=\alpha_p+1}^{\beta_p - 1} (-\partial_{t_2} - q), \quad f_0(t) = \frac{1}{2\pi\iota} \int_{C_1} \Gamma(-\alpha_1 - s)\Gamma(-\alpha_2 - s) t_2^s K_{-s}(t_1)\, ds.$$

Then $f^K_1 = S_3 f_0$, so we show that f_0 is a rapidly decreasing function. Recall an integral formula

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \exp\left(-u - \frac{z^2}{4u}\right) u^{-\nu-1} du$$

of $K_\nu(z)$. Then f_0 is

$$\begin{aligned} &\frac{1}{2\pi\iota} \int_{C_1} \Gamma(-\alpha_1 - s)\Gamma(-\alpha_2 - s) t_2^s \left(\frac{1}{2}\left(\frac{t_1}{2}\right)^{-s} \int_0^\infty \exp\left(-u - \frac{t_1^2}{4u}\right) u^{s-1} du \right) ds \\ &= \int_0^\infty \exp\left(-u - \frac{t_1^2}{4u}\right) \left(\frac{1}{4\pi\iota} \int_{C_1} \Gamma(-\alpha_1 - s)\Gamma(-\alpha_2 - s) \left(\frac{2t_2 u}{t_1}\right)^s ds \right) \frac{du}{u}. \end{aligned}$$

By residue calculus, the inner integral is expressed by a K-Bessel function, and then we get

$$f_0 = \int_0^\infty \exp\left(-u - \frac{t_1^2}{4u}\right) \left(\frac{2t_2 u}{t_1}\right)^{-(\alpha_1+\alpha_2)/2} K_{\alpha_1 - \alpha_2}\left(2\sqrt{\frac{2t_2 u}{t_1}}\right) \frac{du}{u}.$$

This is essentially the same as the function $h_{r,0}$ treated in [6, Theorem 4.4]. This function is proved to be a rapidly decreasing function there. It is not hard to see

that this function generates a rapidly decreasing solution of $\mathcal{D}_{\tilde{\lambda},\eta}\phi = 0$. Therefore, the intertwining operator corresponding to this solution is an element of $\mathrm{Hom}^{\infty}_{G_{\mathbb{R}}}((\pi^*_\Lambda)_\infty, C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))$. This result, together with Proposition 5.14, implies the following theorem.

Theorem 5.15. *Suppose $G_{\mathbb{R}} = Spin(2n,2)$. Let π_Λ be the discrete series representation with the Harish-Chandra parameter $\Lambda \in \Xi_{m,\pm}$, $m = 2,\ldots,n$. If $\mp\eta_2 > 0$, then*

$$\begin{aligned}&\dim\mathrm{Hom}^{\infty}_{G_{\mathbb{R}}}((\pi^*_\Lambda)_\infty, C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta))\\ &\quad = \sum_{\substack{\lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\lambda_{m-2}\geq\mu_{m-2}\geq\lambda_{m-1}\\ \lambda_m\geq\mu'_1\geq\lambda_{m+1}\geq\cdots\geq\lambda_{n-1}\geq\mu'_{n-m}\geq|\lambda_n|}} \dim V^{Spin(2n-3,\mathbb{C})}_{(\mu_1,\ldots,\mu_{m-2},\mu'_1,\ldots,\mu'_{n-m})}.\end{aligned}$$

Each continuous intertwining operator corresponds to f^K_1 defined in Theorem 5.9. On the other hand, if $\mp\eta_2 < 0$, then

$$\mathrm{Hom}^{\infty}_{G_{\mathbb{R}}}((\pi^*_\Lambda)_\infty, C^\infty(G_{\mathbb{R}}/N_{\mathbb{R}};\eta)) = \{0\}.$$

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